Numerical Methods of Homogenization

Week 13–14

November 16 – 24, 2015
Numerical Methods of Homogenization

Goal:
To describe how homogenization theory can be applied to numerical computation of PDEs in highly heterogeneous media

For simplicity, consider the diffusion model in a periodic domain $\Omega$ with period $Y \varepsilon$, where $\varepsilon \ll |\Omega|$ and $Y = [0, 1]^d$

\[
\begin{cases}
-\nabla \cdot (A_\varepsilon(x) \nabla u_\varepsilon(x)) = f(x), & x \in \Omega \\
 u_\varepsilon(x) = 0, & x \in \partial \Omega
\end{cases}
\]

with $f \in L^2(\Omega)$, and $A_\varepsilon(x) = A \left( \frac{x}{\varepsilon} \right)$, where $A(x)$ is $Y$-periodic function and s.t.

\[
\exists \alpha, \beta > 0 : \alpha |\zeta|^2 \leq A_{ij} \left( \frac{x}{\varepsilon} \right) \zeta_i \zeta_j \leq \beta |\zeta|^2, \quad \forall \zeta \in \mathbb{R}^d, \quad \forall x \in \Omega
\]

Remark:
Any method for numerical approximation of solution $u_\varepsilon$ to (1) requires a mesh size $h \ll \varepsilon \ll 1$. If $\varepsilon$ is too small, the method yields a very fine mesh and, thus, a very large number of degrees of freedom
The classical approach is to numerically solve (1) is rather to compute the solution of homogenized equation corresponding to (1), that is,

\[
\begin{cases}
-\nabla \cdot [A^* \nabla u(x)] = f(x), & x \in \Omega \\
u(x) = 0, & x \in \partial \Omega
\end{cases}
\]

(2)

with effective conductivity

\[A_{ij}^* = \int_Y A(y) [e_j + \nabla \lambda_j(y)] \cdot e_i \, dy\]

(3)

where \(\lambda_j(y)\) solves the cell problem:

\[j \in \{1, \ldots, d\} : \begin{cases}
-\nabla_y \cdot [A(y) \nabla_y \lambda_j(y)] = -\nabla_y \cdot [A(y)e_j], & y \in Y \\
\lambda_j(y) \text{ is } Y - \text{periodic}, & \int_Y \lambda_j(y) dy = 0
\end{cases}\]

(4)
An additional advantage of the cell problem (4) is that it allows to improve the approximation of $u_\varepsilon(x)$ by $u(x)$ adding the so-called corrector to the homogenized solution:

$$u_\varepsilon(x) \approx u(x) + \varepsilon \sum_{j=1}^{d} \lambda_j \left( \frac{x}{\varepsilon} \right) \frac{\partial u}{\partial x_j}(x)$$  \hspace{1cm} (5)

The of the gradient corrector term of (5) is of the same order as the homogenized gradient $\nabla u(x)$:

$$\nabla u_\varepsilon(x) \approx \nabla u(x) + \sum_{j=1}^{d} \nabla_y \lambda_j \left( \frac{x}{\varepsilon} \right) \frac{\partial u}{\partial x_j}(x)$$

Also, remark that if $A(y)$ has a cubic symmetry in $Y$, i.e. is a diagonal matrix, symmetric w.r.t. hyperplanes, parallel to the faces of $Y$ funning through its center, then it is easy to check that

$\lambda_i(y)$ is even w.r.t. $y_j$ for $j \neq i$, and $\lambda_i(y)$ is odd w.r.t. $y_i$

With that, denote by $\Gamma_i$ the faces of $Y$ that are normal to $e_i$, then (4) is equivalent to

$$\begin{cases} 
-\nabla_y \cdot [A(y) (\nabla_y \lambda_i(y) + e_i)] = 0, & y \in Y \\
\lambda_i(y) = 0, & y \in \Gamma_i \\
\frac{\partial \lambda_i}{\partial n}(y) = 0, & y \in \Gamma_j, j \neq i
\end{cases}$$  \hspace{1cm} (6)
Heterogeneous, non-periodic media

Goal:
To devise a numerical method with a mesh size $H \gg \varepsilon$, which is not a mere computations of the homogenized solution of (2) but takes into account its microscopic fluctuations.

Main Idea:
Instead of using the standard, say $P_1$, finite element (FE) basis, we first construct oscillating FE basis.

Start with building a coarse mesh $\mathcal{T}_H = \{ K \}$ of the domain $\Omega$ of size $H \gg \varepsilon$ with vertices $\{ x_i \}_{i=1,\ldots,N_H}$, where $N_H$ is the total number of vertices of the coarse mesh.

Then we construct a special FE basis adapted to the problem. Each mesh cell $K$ has its own fine mesh, independent from the other ones.
For each mesh cell $K$ and each its vertex $x_i$ compute a basis function $\varphi_{i,K}^\epsilon$ as a solution of

\[
\begin{cases}
-\nabla \cdot (A^\epsilon(x) \nabla \varphi_{i,K}^\epsilon) = 0, & \text{in } K \\
\varphi_{i,K}^\epsilon = \delta_{ij}, & \text{at vertex } x_i \\
\varphi_{i,K}^\epsilon & \text{is affine on } \partial K
\end{cases}
\]  

(7)

The BVP (7) is similar to the cell problem (4). Indeed, consider an affine function $\mathbf{e} \cdot \mathbf{x} + c$ that coincides with BC of (7), that is, $\mathbf{e} \cdot x_j + c = \delta_{ij}$ and definining

$$\lambda_{i,K}^\epsilon = \varphi_{i,K}^\epsilon - \mathbf{e} \cdot \mathbf{x} - c,$$

then (7) is equivalent to

\[
\begin{cases}
-\nabla \cdot \left[ A^\epsilon(x) \left( \mathbf{e} + \nabla \lambda_{i,K}^\epsilon \right) \right] = 0, & \text{in } K \\
\lambda_{i,K}^\epsilon = 0, & \text{on } \partial K
\end{cases}
\]  

(8)

The main difference between (4) and (8) is in the BC: the periodic for the former one and the Dirichlet for the latter.
Collecting these functions $\varphi_{i,K}^\varepsilon$ for all cells $K$ around a single vertex $x_i$, we obtain a basis function $\varphi_i^\varepsilon$ with a compact support.

Finally, it remains to compute an approximation $u_H^\varepsilon$ of the true solution $u_\varepsilon$ of (1) by using the FE basis $\{\varphi_i^\varepsilon\}_{1 \leq i \leq N_H}$ on the coarse mesh $T_H$.

This last problem is of moderate size, hence, of low cost, and it incorporates the oscillations of the heterogeneous tensor $A_\varepsilon$.

In the periodic case the convergence result is

$$\exists C > 0 : \text{ s.t. } \|u_H^\varepsilon - u_\varepsilon\|_{H^1_0(\Omega)} \leq C \left( H + \sqrt{\frac{\varepsilon}{H}} \right)$$

Clearly, this estimate is interesting only if $H \gg \varepsilon$. 

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Oversampling

There is a version of the above method called Multiscale Finite Element Method (MsFEM) with Oversampling that improves estimate (9) by eliminating the $\sqrt{\cdot}$-term.

This generalization amounts to compute the basis functions $\varphi_{i,K}^\varepsilon$ on a coarse mesh $K'$ which is slightly larger than $K$, namely, $K' \subset\subset K$.

With that, this method becomes non-conforming unlike the above one, since the different functions $\varphi_{i,K}^\varepsilon$ do not match at the interface between the neighboring cells. However, this generalization suppresses the boundary effects near the cell boundaries $\partial K$. 
References


