Adjoint, Unitary, Normal, Self-Adjoint Operators

Week 1

January 20–23, 2015
Adjoint Operator

Let $X$ and $Y$ be normed, $A \in \mathcal{B}(X, Y)$. Consider

\[ \langle Ax, f \rangle \text{ where } x \in X, f \in X^* \]

Define a functional

\[ \langle x, \varphi \rangle = \langle Ax, f \rangle, \quad x \in X \]

Then

1. $D(\varphi) = X$
2. $\varphi$ is linear:

\[
\langle \alpha_1 x_1 + \alpha_2 x_2, \varphi \rangle = \langle A(\alpha_1 x_1 + \alpha_2 x_2), f \rangle = \alpha_1 \langle Ax_1, f \rangle + \alpha_2 \langle Ax_2, f \rangle
\]

\[ = \alpha_1 \langle x_1, \varphi \rangle + \alpha_2 \langle x_2, \varphi \rangle \]

3. $\varphi$ is bounded:

\[
|\varphi(x)| = |\langle Ax, f \rangle| \leq \|Ax\| \|f\| \leq \|A\| \|f\| \|x\| \leq C \|x\|, \quad \forall x \in X
\]

\[ \Rightarrow \varphi \in X^* \]
Adjoint Operator (cont.)

Hence, for each \( f \in Y^* \) we have constructed through \( \langle x, \varphi \rangle = \langle Ax, f \rangle \) an element \( \varphi \in X^* \), i.e. constructed a linear bounded operator \( A^* : Y^* \rightarrow X^* \) where \( A^* \in \mathcal{B}(Y^*, X^*) \) and \( \varphi = A^*f \) (\( f \in Y^* \), \( \varphi \in X^* \))

**Definition (adjoint operator)**

The operator \( A^* \) s.t. \( \langle x, A^*f \rangle = \langle Ax, f \rangle \), is called the adjoint to \( A \)

**Definition (adjoint operator in a Hilbert space)**

For a Hilbert space \( H \), consider \( A \in \mathcal{B}(H) \) then the adjoint \( A^* \) of \( A \) is the mapping \( A^* : H \rightarrow H \) defined by \( A^*y = z_y \), \( y \in H \) where \( z_y \in H \) is that unique element s.t. \( (Ax, y) = (x, z_y) \)

**Theorem**

\( A \in \mathcal{B}(X, Y) \) then \( \|A^*\| = \|A\| \)
Adjoint Operator (cont.)

**Examples:**

1. \( X = Y = \mathbb{R}^n \). Consider a linear operator \( A : X \to Y \) defined by \( y = Ax \), where \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) is a matrix, \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) via

\[
y_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, n
\]

then let \( z \in (\mathbb{R}^n)^* = \mathbb{R}^n \):

\[
\langle Ax, z \rangle = (Ax, z) = \sum_{i=1}^{n} y_i z_i = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} x_j \right) z_i = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} a_{ij} z_i \right) x_j = (x, A^* z) = \langle x, A^* z \rangle
\]

where the operator \( A^* \) defined \( w = A^* z \) is defined as follows:

\[
w_j = \sum_{i=1}^{n} y_i z_i, \quad j = 1, \ldots, n
\]

i.e. \( A^* \) is defined by the matrix \( A^T \) (in \( \mathbb{C}^n \): \( A^* = \overline{A^T} \))
Adjoint Operator (cont.)

Examples:

2. \( X = Y = L_2[a, b] \). Consider an integral operator \( \mathcal{K} \) defined by

\[
y(t) = \int_a^b K(t, s)x(s) \, ds
\]

with continuous in \([a, b] \times [a, b]\) kernel \( K(t, s) \). Then \( z \in L_2[a, b] \)

\[
\langle \mathcal{K}x, z \rangle = \int_a^b (K(t, s)x(s) \, ds)z(t) \, dt
\]

\[
= \int_a^b (K(t, s)z(t) \, dt)x(s) \, ds = \langle x, \mathcal{K}^* z \rangle
\]

then the adjoint operator \( w = \mathcal{K}^* z \) also an integral operator

\[
w(t) = \int_a^b K(s, t)z(s) \, ds
\]
Adjoint Operator (cont.)

**Theorem**

Let $A \in \mathcal{B}(H)$ and $\lambda \in \mathbb{K} = \{\mathbb{R}, \mathbb{C}\}$. Then

1. $A^{**} := (A^*)^* = A$
2. $(\lambda A)^* = \overline{\lambda} A^*$
3. $(A + B)^* = A^* + B^*$
4. $(AB)^* = B^* A^*$
5. *If $A$ is invertible then so is $A^*$ and $(A^*)^{-1} = (A^{-1})^*$*
6. $\|A^* A\| = \|A\|^2$

**Definition (self-adjoint, unitary, normal operators)**

Let $H$ be a Hilbert space over $\mathbb{K} = \{\mathbb{R}, \mathbb{C}\}$. An operator $A \in \mathcal{B}(H)$ is called:

1. **self-adjoint** (or hermitian) iff $A^* = A$, i.e.
   \[(Ax, y) = (x, Ay), \quad \forall x, y \in H\]
2. **unitary** (or orthogonal if $\mathbb{K} = \mathbb{R}$) iff $A^* A = AA^* = I$
3. **normal** iff $A^* A = AA^*$

Obviously, self-adjoint and unitary operators are normal.
Examples:

1. $A = 2il \Rightarrow A^* A = (2il)^*(2il) = -2il \cdot 2il = 4I = (2il)(-2il) = (2il)(2il)^* = AA^*$

2. $A_r : \ell_2 \to \ell_2$ with $(x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, x_3, \ldots)$ (right shift) and $A_\ell : \ell_2 \to \ell_2$ with $(x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$ (left shift)

   $$(A_r x, y) = ((0, x_1, x_2, \ldots)(y_1, y_2, \ldots)) = \sum_{j=1}^{\infty} x_j y_{j+1}$$

   $$= ((x_1, x_2, x_3, \ldots)(y_2, y_3, \ldots)) = (x, A_\ell y)$$

   $\Rightarrow A_r^* = A_\ell$ (i.e. not self-adjoint) $A_r^* A_r = A_\ell A_r = I$

   However,

   $$A_r A_r^* (x_1, x_2, x_3, \ldots) = A_r A_\ell (x_1, x_2, x_3, \ldots)$$

   $$= A_r (x_2, x_3, \ldots) = (0, x_2, x_3, \ldots) \neq I$$

   $\Rightarrow A_r^* A_r \neq A_r A_r^*$ (i.e. not unitary, nor normal)

3. In $\mathbb{R}^n$, $A$ is self-adjoint iff $a_{ij} = a_{ji} \Rightarrow A$ is a symmetric matrix

   In $L_2(a, b)$, an integral operator $\mathcal{K}$ is self-adjoint iff its kernel is symmetric, i.e. $K(t, s) = K(s, t)$
Adjoint Operator (cont.)

Theorem (1)

Let $A, B$ be self-adjoint in $H$, $\alpha, \beta \in \mathbb{R} \Rightarrow \alpha A + \beta B$ is also self-adjoint

Theorem (2)

Let $H$ be a complex Hilbert space and $T, S \in \mathcal{B}(H)$. Then

1. $(Tx, x = 0) \forall x \in H \Rightarrow T = 0$
2. $(Tx, x = 0) = (Sx, x) \forall x \in H \Rightarrow T = S$

Theorem (3)

Let $H$ be a Hilbert space over $\mathbb{K} = \{\mathbb{R}, \mathbb{C}\}$ and $A \in \mathcal{B}(H)$. Then

1. If $A$ is self-adjoint in $H \Rightarrow (Ax, x) \in \mathbb{R}$
2. If $\mathbb{K} = \mathbb{C}$ and $(Ax, x) \in \mathbb{R} \forall x \in H \Rightarrow A$ is self-adjoint in $H$
### Definition (quadratic form)

\((Ax, x)\) is called a **quadratic form** of \(A\)

### Theorem (4)

Let \(H\) be a Hilbert space over \(\mathbb{K} = \{\mathbb{R}, \mathbb{C}\}\) and \(A \in \mathcal{B}(H)\) be self-adjoint. Then

\[
\|A\| = \sup_{\|x\|=1} |(Ax, x)|
\]

### Theorem (5)

Let \(A : H \to H\) be a linear bounded operator. Then

\[
\overline{\mathcal{R}(A)} = (\ker A^*)^\perp, \quad \ker A = (\mathcal{R}(A^*))^\perp
\]
Note 1:
An equivalent statement of THM5 is that if $A \in \mathcal{B}(H)$ then
\[ H = \overline{\mathcal{R}(A)} \oplus \ker A^* \]

Note 2:
The adjoint $A^*$ plays a crucial role in studying solvability of a linear equation
\[ Ax = y, \quad A : H \to H, \quad A \in \mathcal{B}(H) \quad (1) \]
Let $z \in H$ be any solution of the homogeneous adjoint equation
\[ A^* z = 0, \quad \text{(i.e. } z \in \ker A^*) \quad (2) \]
then taking the inner product of (1) with $z$:
\[ (Ax, z) = (y, z) = 0 = (x, A^* z) \]
\[ \Rightarrow \text{ a necessary condition for a solution of (1) to exist is} \]
\[ (y, z) = 0 \quad \forall \ z \in \ker A^*, \quad \text{i.e. } y \in (\ker A^*)^\perp \]
References