Introduction. Classification of DEs. Solution to DE

Week 1

August 21–25, 2017
Can be downloaded at course webpage

1 Course Info:
   - Meet T/Th 11:30am–1:00pm in SEC 206

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4 Assessment:

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<th>HW</th>
<th>Midterm I</th>
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<td>40%</td>
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<tr>
<td>Every week</td>
<td>October 5, 2017</td>
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Syllabus (cont.)

5 HOMEWORK
- Usually assigned every week and are due on Tuesday of the next week in the lecture
- Do all the assigned homework problems, however only a few randomly selected problems will be graded
- Late homework will not be accepted
- Collaboration on homework is allowed, but you have to write your entire solution by yourself
- All work must be shown
- Do not forget to put your name and ID # on it, homework number and staple all the sheets
- All homework assignments are mandatory, although, only 10 the best scores would count in the final grade

6 EXAMS:
- Class midterms
- There is no final exam
- No collaboration on exams is allowed
- No books, cell phones, calculators or notes of any sort
# ODEs: Introduction

## Definition
An equation that contains a derivative of an unknown function is called **differential equation** (DE).

## Definition
**Ordinary differential equation** (ODE) contains an ordinary derivative.

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms the relations are **equations** and the rates are **derivatives**.

## Definition
A differential equation that describes some physical process is often called **a mathematical model** of the process.
# Classification of DEs

1. Based on the number of independent variables the unknown function depends on: **ODE vs PDE (Partial differential equation)**

   **Examples:**
   - $t^2 y' + 5(\sin t)y = -2 \rightsquigarrow \text{ODE for } y = y(t)$
   - $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = f(x, y) \rightsquigarrow \text{PDE for } u = u(x, y)$

2. Based on the order of the highest derivative of unknown fn: **Order of DE**

   **Definition**
   - The order of a DE is the order of the highest derivative of unknown function

   Generally, the equation
   
   $F[t, y(t), y'(t), \ldots, y^{(n)}(t)] = 0$  \hspace{1cm} (1)

   is an ordinary differential equation of the $n^{\text{th}}$ order.

   **Examples:**
   - $u''' + 3u = \cos x \rightsquigarrow 3\text{rd order ODE for } u = u(x)$
   - $\frac{d^2 y}{dx^2} = 0 \rightsquigarrow 2\text{nd order ODE for } y = y(x)$
Classification of DEs (cont.)

3 Linear vs Nonlinear DE

Definition
An ODE is **linear** is the function $F$ in (1) is linear,

that is, the general linear ODE of order $n$ is

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = g(t)$$  \hspace{1cm} (2)

Most of the equations that we discuss in this course are linear

Definition
An ODE that is not of the form (2) is a **nonlinear equation**

Examples:
- $x^3 y''' + 2(\cos x)y' = -\frac{1}{1 + x^2} \quad \rightsquigarrow \text{linear ODE}$
- $u'' + 2uu' = 0 \quad \rightsquigarrow \text{nonlinear ODE (because of the term } 2uu')$
- $(y')^2 + 3t^2y' - 5y = 0 \quad \rightsquigarrow \text{nonlinear ODE (because of } (y')^2)$
Classification of DEs (cont.)

Based on the number of unknown functions:

Single Equation vs System of DEs

Examples:

- $x^3 y''' + 2(\cos x)y' = -\frac{1}{1 + x^2}$ \implies single ODE for $y = y(x)$

- $\begin{cases} u' + 2v = t \\ v' - u = 0 \end{cases}$ \implies system of ODEs for $u = u(t)$ and $v = v(t)$
A solution of a DE is a differentiable function that gives a trivial identity after being substituted back to the equation.

Verify that \( y(t) = 3t + t^2 \) is the solution of \( ty' - y = t^2 \)?

Evaluate: \( y'(t) = 3 + 2t \) and substitute back to the equation:
\[ t(3 + 2t) - 3t - 2 = t^2 = \text{LHS}^†, \text{ whereas } RHS = t^2 \implies \text{LHS} \equiv \text{RHS} \]

† LHS = “Left hand side”, and RHS = “Right hand side”

A family of solutions to a given ODE is called general solution that contains constant(s) coming from integration.

To obtain a representative of this family of solutions, one needs to prescribe an initial condition. For ex., for the 1st order ODE we have

Initial Value Problem (IVP):
\[
\begin{align*}
y'(t) &= f(t, y) \\
y(t_0) &= y_0
\end{align*}
\]
where \( t_0 \) and \( y_0 \) are given numbers.
Falling Object Example

Set up and solve a mathematical model for an object that is falling in the atmosphere near sea level from the rest.

- $t \rightsquigarrow$ time, [s] (independent variable)
- $v \rightsquigarrow$ velocity, $v = v(t)$, [m/s] (dependent variable)
- $m \rightsquigarrow$ mass, [kg]
- $g \rightsquigarrow$ acceleration due to gravity, $g = 9.8$ [m/s$^2$]
- $\gamma \rightsquigarrow$ drag coefficient, [kg/s]
- $a \rightsquigarrow$ acceleration, [m/s$^2$]
- $F \rightsquigarrow$ net force, [N = kg × m/s$^2$]

Newton’s 2nd Law:

$$\vec{F} = m\vec{a}$$

Note, that both $\vec{F}$ and $\vec{a}$ are vectors so we need into account their directions. The net force $\vec{F}$ consists of the gravity $\vec{f}_{\text{grav}}$ and air resistance force (drag) $\vec{f}_{\text{drag}}$ so that $F = \vec{f}_{\text{grav}} - \vec{f}_{\text{drag}}$, where $\vec{f}_{\text{grav}} = mg$, $\vec{g} = (0, g)^T$. The drag force is somewhat difficult to model but we assume that it is proportional to the velocity $v$, i.e. $f_{\text{drag}} = \gamma v(t)$.

Lastly, recall that the acceleration is $\vec{a} = \frac{d\vec{v}}{dt}$.
Hence, the equation of motion of the object becomes:

\[ m \frac{dv}{dt} = mg - \gamma v \]

To conclude setting up the model we have to supply the above ODE with an initial condition (IC), which is \( v(0) = 0 \), as the object started its motion from rest. Therefore,

**Mathematical Model:**

\[
\begin{align*}
    m \frac{dv}{dt} &= mg - \gamma v \\
    v(0) &= 0
\end{align*}
\]  

Now assume, that \( m = 10 \text{ kg} \) and \( \gamma = 2 \text{ kg/s} \), then the ODE in (3) becomes

\[ \frac{dv}{dt} = 9.8 - \frac{v}{5}, \]

that we will solve. For that, rewrite \( \frac{dv}{dt} = -\frac{v - 49}{5} \implies \frac{dv}{v - 49} = -\frac{1}{5} \).
We now observe, that

\[
\frac{dv}{dt} = \frac{d}{dt} (\log |v - 49|),
\]

then \( \frac{d}{dt} (\log |v - 49|) = -\frac{1}{5} \).

Now integrate: \( \log |v - 49| = -\frac{1}{5} t + C \), that we solve and obtain:

\[
v(t) = 49 + Ce^{-t/5}, \quad \text{where } C \text{ is an arbitrary const} \tag{4}
\]

The function \( v \) in (4) is the general solution, and we have to find a particular solution satisfying the given IC \( C = -49 \):

\[
v(t) = 49 - 49e^{-t/5} [\text{m/s}], \tag{5}
\]

that gives dependence of the object's velocity on any moment of time \( t \).