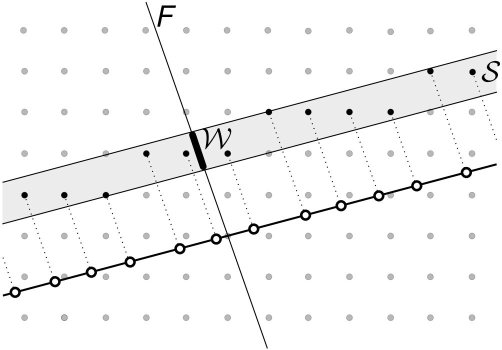
Introduction to mathematical quasicrystals



Alan Haynes

Topics to be covered

- Historical overview: aperiodic tilings of Euclidean space and quasicrystals
- Lattices, crystallographic point sets, and cut and project sets in Euclidean space
- Rotational symmetries, crystallographic restriction theorem
- Diffraction
- Complexity and repetitivity of patches

§1 Historical overview: quasicrystals and aperiodic tilings of Euclidean space

Physical quasicrystals

- A physical crystal is a material whose atoms or molecules are arranged in a highly order way.
- Crystallographic Restriction Theorem (Haüy, 1822): Rotational symmetries in the diffraction patterns of (periodic) crystals are limited to $1, 2, 3, 4$, and 6-fold.

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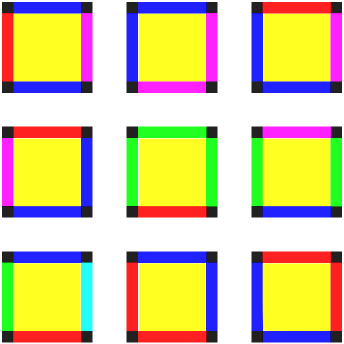
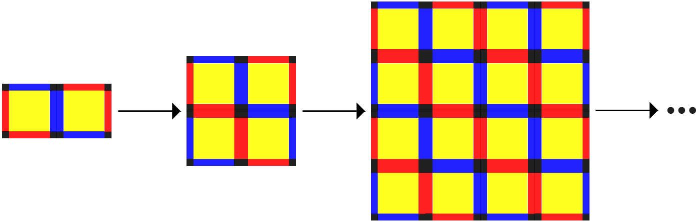
- Crystallographic Restriction Theorem (Haüy, 1822): Rotational symmetries in the diffraction patterns of (periodic) crystals are limited to $1, 2, 3, 4$, and 6-fold.
- Shechtman (1982): Discovered crystallographic materials with diffraction exhibiting 10-fold symmetry.
- The 'forbidden symmetries' observed in quasicrystals are possible because they lack translational symmetry.

Wang tiles and the domino problem (1960's)



Example of a Wang tiling



The Domino Problem

- Is there an algorithm which, when given any finite collection of Wang tiles, can decide whether or not it can tile the plane?
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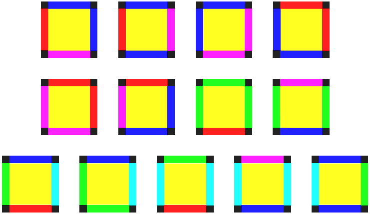
- Berger (1966) answered the domino problem in the negative, by relating it to the halting problem for Turing machines.
- Berger also came up with an explicit example of a collection of $20,426$ Wang tiles which can tile the plane, but only aperiodically.

Aperiodic sets of prototiles

- More recently, an argument due to Kari and Culik (1996), led to discovery of the following set of Wang tiles:



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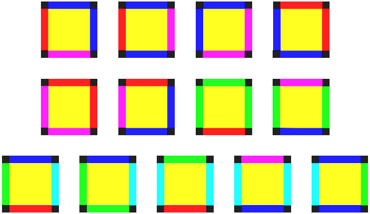
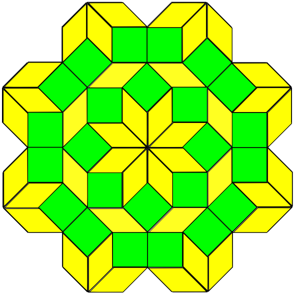
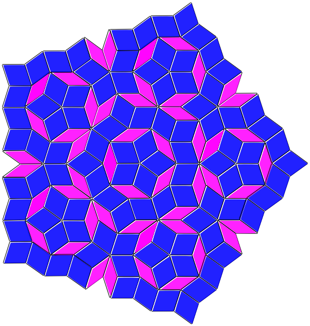


- In (2015), Emmanuel Jeandel and Michael Rao found a set of 11 Wang tiles with 4 colors which tile the plane only aperiodically, and they proved that this is both the minimum possible number of tiles, and of colors for such a tiling.

Aperiodic tilings of Euclidean space





Three methods for tiling Euclidean space

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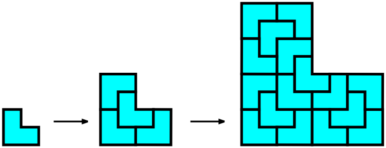
- Substitution rules: Start with a finite collection of prototiles tiles and a rule for inflating them, and then partitioning the inflated tiles back into prototiles.



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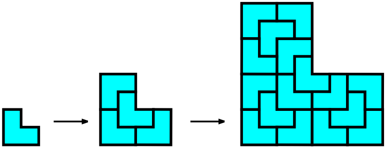


- Cut and project method: A dynamical method which projects a slice of a higher dimensional lattice to a lower dimensional space.

§2 Point sets in Euclidean space

Definitions and terminology

- A countable subset of $\mathbb{R}^k$ is called a **point set.**
- $Y \subseteq \mathbb{R}^k$ is **uniformly discrete** if there is a constant $r > 0$ such that, for all $y \in Y$,

$$B_r(y) \cap Y = \{y\}.$$

- $Y \subseteq \mathbb{R}^k$ is **relatively dense** if there is a constant $R > 0$ such that, for any $x \in \mathbb{R}^k$,

$$\overline{B_R(x)} \cap Y \neq \emptyset.$$

- A set $Y \subseteq \mathbb{R}^k$ which is both uniformly discrete and relatively dense is called a **Delone set.**

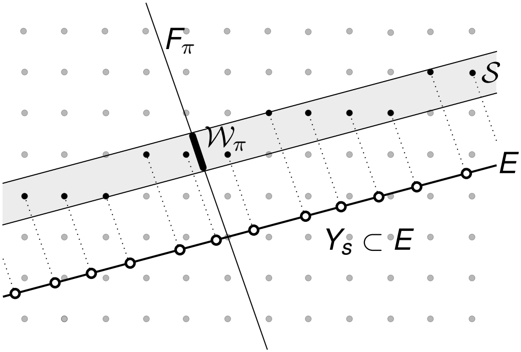
First examples of Delone sets

- A **lattice** in $\mathbb{R}^k$ is a discrete subgroup $\Lambda \leqslant \mathbb{R}^k$ with the property that the quotient space $\mathbb{R}^k/\Lambda$ has a Lebesgue measurable fundamental domain with finite volume.
- A set $Y \subseteq \mathbb{R}^k$ is called a **crystallographic point set** if it can be written as
$$Y = \Lambda + F,$$
where Λ is a lattice in $\mathbb{R}^k$ and $F \subseteq \mathbb{R}^k$ is a finite set.

Groups of periods

- If $Y \subseteq \mathbb{R}^k$ is a point set, then a point $x \in \mathbb{R}^k$ with the property that $Y + x = Y$ is called a **period** of Y. The collection of all periods of Y forms a group, called its **group of periods**.
- We say that Y is **nonperiodic** if its group of periods is $\{0\}$, and we say that Y is **periodic** otherwise.
- Lemma: A uniformly discrete point set $Y \subseteq \mathbb{R}^k$ is a crystallographic point set if and only if its group of periods is a lattice in $\mathbb{R}^k$.

§3 Cut and project sets



Cut and project sets: definition

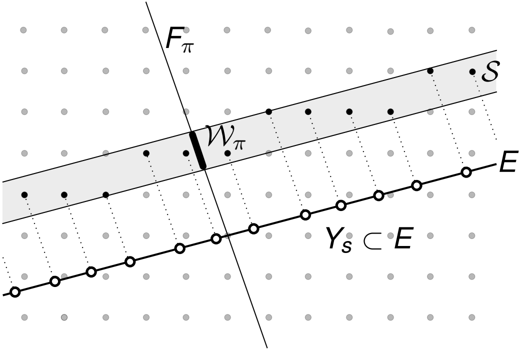
For $k > d \geq 1$, start with the following data:

- Subspaces E and F_π of $\mathbb{R}^k$, $\dim(E) = d$, $E \cap F_\pi = \{0\}$, and
$$\mathbb{R}^k = E + F_\pi,$$
- Natural projections π and π^* from $\mathbb{R}^k$ onto E and F_π,
- A subset $\mathcal{W}_\pi \subseteq F_\pi$, called the **window**,
- A point $s \in \mathbb{R}^k$.

The **k to d cut and project set** defined by this data is:

$$Y_s = \pi\{n + s : n \in \mathbb{Z}^k, \pi^*(n+s) \in \mathcal{W}_\pi\}.$$

Cut and project sets: terminology



$\mathbb{R}^k$: **total space**

E : **physical space**

F_π : **internal space**

$\mathcal{W}_\pi$: **window**

$\mathcal{S}$: **strip**

$$Y_s = \pi\{n + s : n \in \mathbb{Z}^k, \pi^*(n+s) \in \mathcal{W}_\pi\} = \pi(\mathcal{S} \cap (\mathbb{Z}^k + s)).$$

Example: 2 to 1 cut and project set

- Consider the subspace E of $\mathbb{R}^2$ generated by the vector

$$\begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix},$$

- $F_\pi = E^\perp$, and W_π is the image under π^* of the vertical interval

$$\{(0, x_2) : 2 - \sqrt{5} \leq x_2 < (3 - \sqrt{5})/2\} \subseteq \mathbb{R}^2.$$

Fibonacci tiling

$$a \mapsto ab \qquad b \mapsto a$$

$$a \mapsto ab \mapsto aba \mapsto abaab \mapsto \cdots$$

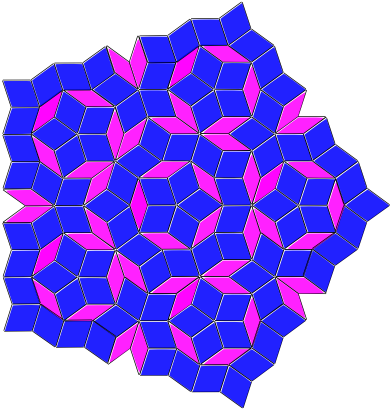
Example: 5 to 2 cut and project set

- Consider the subspace E of $\mathbb{R}^5$ generated by the columns of the matrix

$$\begin{pmatrix} 1 & 0 \\ \cos(2\pi/5) & \sin(2\pi/5) \\ \cos(4\pi/5) & \sin(4\pi/5) \\ \cos(6\pi/5) & \sin(6\pi/5) \\ \cos(8\pi/5) & \sin(8\pi/5) \end{pmatrix},$$

- F_π chosen appropriately, and $\mathcal{W}_\pi$ the **canonical window**, which is the image under π^* of the unit cube in $\mathbb{R}^5$.

Penrose tiling



What we will always assume

(i) $\mathcal{W}_\pi$ is bounded and has nonempty interior, and the closure of $\mathcal{W}_\pi$ equals the closure of its interior

(ii) $\pi|_{\mathbb{Z}^k}$ is injective

(iii) $s \notin (\mathbb{Z}^k + \partial\mathcal{S})$ (Y_s is **nonsingular**)

What we will usually assume

(iv) $E + \mathbb{Z}^k$ is dense in $\mathbb{R}^k$ (E **acts minimally** on $\mathbb{T}^k$)

(v) If $p + Y = Y$ then $p = 0$ (Y is **aperiodic**)

(vi) E can be parametrized as

$$E = \{(x_1, \ldots, x_d, L_1(x), \ldots, L_{k-d}(x)) : x \in \mathbb{R}^d\}$$

A couple of remarks:

- Assumptions (i)+(iv) guarantee that Y is a Delone set.
- Neither the truth of condition (iv) nor that of (v) implies the other.

One consequence

Assumptions (i)+(v) guarantee that Y is a **Delone set**:

- **uniformly discrete**: $\exists r > 0$ such that, for any $y \in Y$,

$$Y \cap B_r(y) = \{y\},$$

- **relatively dense**: $\exists R > 0$ such that, for any $x \in E$,

$$Y \cap \overline{B_R(x)} \neq \emptyset.$$

Reference subspace

As a reference point, when allowing E to vary, we also make use of the fixed $(k-d)$-dimensional subspace F_ρ of $\mathbb{R}^k$ defined by

$$F_\rho = \{(0, \ldots, 0, y) : y \in \mathbb{R}^{k-d}\}$$

and we let $\rho : \mathbb{R}^k \to E$ and $\rho^* : \mathbb{R}^k \to F_\rho$ be the projections onto E and F_ρ with respect to the decomposition

$$\mathbb{R}^k = E + F_\rho.$$

We set

$$\mathcal{W} = \rho^*(\mathcal{W}_\pi),$$

and we also refer to this set as the window.

Two special types of windows

- The **cubical** window,

$$\mathcal{W} = \left\{ \sum_{i=d+1}^{k} t_i e_i : 0 \leq t_i < 1 \right\}.$$

- The **canonical** window,

$$\mathcal{W} = \rho^* \left(\left\{ \sum_{i=1}^{k} t_i e_i : 0 \leq t_i < 1 \right\} \right).$$

We say that Y is a **cubical** (resp. **canonical**) **cut and project set** if it is nonsingular, minimal, and aperiodic, and if $\mathcal{W}$ is a cubical (resp. canonical) window.

§4 Crystallographic restriction and rotational symmetry

Rotations and *n*-fold symmetry

- Identify the **group of rotations** of $\mathbb{R}^k$ with the special orthogonal group $\mathrm{SO}_k(\mathbb{R})$, the group of $k \times k$ orthogonal matrices with determinant 1.
- A point set $Y \in \mathbb{R}^k$ has **n-fold symmetry** if there is an element $A \in \mathrm{SO}_k(\mathbb{R})$ of order n which stabilizes Y (i.e. such that that $AY = Y$).
- A rotation $A \in \mathrm{SO}_k(\mathbb{R})$ is an **irreducible rotation of order** n if $A^n = \mathrm{I}$ and if, for any $1 \le m < n$ the only element of $\mathbb{R}^k$ which is fixed by A^m is $\{0\}$. If a point set $Y \subseteq \mathbb{R}^k$ is stabilized by an irreducible rotation of $\mathbb{R}^k$ of order n then we say that Y has has **irreducible n-fold symmetry**.

Crystallographic restriction

- Lemma: If a lattice $\Lambda \subseteq \mathbb{R}^k$ has irreducible *n*-fold rotational symmetry, then it must be the case that $\varphi(n)|k$.
- Crystallographic Restriction Theorem: A lattice in 2 or 3 dimensional Euclidean space can have *n*-fold symmetry only if $n = 1, 2, 3, 4$, or 6.

Planar cut and project sets with *n*-fold symmetry

- Lemma: Choose $n \in \mathbb{N}$ and suppose that $\varphi(n)|k$. Then there is a lattice in $\mathbb{R}^k$ with irreducible *n*-fold symmetry.
- Theorem: For any $n > 2$, there is a k to 2 cut and project set, with $k = \varphi(n)$, with *n*-fold rotational symmetry.

Exercises from lecture notes

(3.5.2) Prove the Crystallographic Restriction Theorem above, but for crystallographic point sets instead of lattices.

(3.5.1) Give an example of a lattice $\Lambda \subseteq \mathbb{R}^6$ with 15-fold rotational symmetry.