Topology of tiling spaces 3/4

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Important facts which make Cech cohom. the "right" invariant for us:

1) $\check{H}^k$(CW complex) = H^k(CW complex)

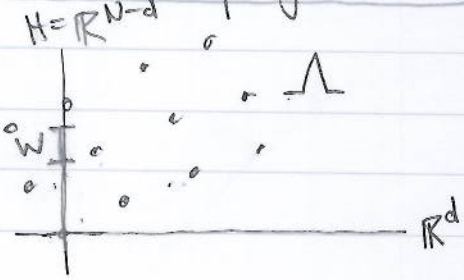
2) $\check{H}^k(\varprojlim(\Gamma^n, \rho_n)) = \varinjlim(\check{H}^k(\Gamma^n))$
   (direct limit)

Review of direct limits:

Given a sequence of groups G^n and homoms. $\rho_n : G^n \to G^{n+1}$, the direct limit $\varinjlim(G^n, \rho_n)$ is the disjoint union of the groups G^n modulo the equiv. relation defined by the rule $x \sim \rho_n(x)$.

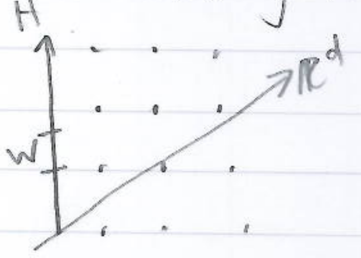
Ex: If $G^n = \mathbb{Z}$ and ρ_n is multiplication by 2 then $\varinjlim(G^n, \rho_n) \cong \mathbb{Z}[1/2]$ (the additive group of rationals with even denoms.)

Cut-and-project tilings



We define two projections $\pi^{\parallel} : H \times \mathbb{R}^d \to \mathbb{R}^d$ and $\pi^{\perp} : H \times \mathbb{R}^d \to H$, so that $\pi^{\parallel}$ is injective.

Note: A viewpoint which is essentially equivalent is to take $\Lambda = \mathbb{Z}^N$ and think of $\mathbb{R}^d$ as meeting H with an "irrational" slope:



Sidenote: We are focusing on point patterns. However if we take the Voronoi cells surrounding the points in such a pattern, and then take the dual tiling, we obtain a tiling whose vertices are exactly the points in the point pattern.

Now $\forall \xi \in \mathbb{R}^N$ we can define $Y_\xi = \pi^{\shortparallel}((\Lambda+\xi) \cap S)$. We call ξ <u>nonsingular</u> if none of the points of $\Lambda+\xi$ fall on ∂S. Choose a nonsingular ξ and consider Ω_{Y_ξ}.

There is a continuous map from Ω_{Y_ξ} to $H \times \mathbb{R}^d / \Lambda$ given by sending a tiling $Y_{\xi'}$ to the parameter ξ'. Now there is an important issue that arises when we consider limit points: If we approach a singular point from different directions, we can get different limit points depending on whether or not the points were contained in the strip, in the limit. This means that topologically, the space of all tilings coming from our cut-and-project setup is a torus, with a set of measure 0 (the projection of ∂S) cut out and glued back in multiple time.

Ex: $N=2$, $d=1$: The space of all tilings we can obtain looks like $\mathbb{T}^2$ with one or two lines cut out (assuming ω is an interval). If the endpts. of ω are related by an elem. of Λ then there is one line cut out, otherwise there are two.

The resulting space can be realized as an inverse limit of punctured tori.

Since the maps in the limit give isomorphisms of cohomology, we can understand the topology of the resulting space by understanding the topology of the approximants.

The general description of topology of cut-and-proj. tilings is given in a book by Forrest, Hunton, and Kellendonk.

<u>PE cohomology</u>

T = tiling, $f : \mathbb{R}^d \to A$ is strongly pattern equivariant (sPE) with radius R if

$(T-x) \cap B_R(0) = (T-y) \cap B_R(0) \Rightarrow f(x) = f(y)$

f is sPE if $\exists R$ s.t. ...

f is weakly PE if $\forall \varepsilon > 0\ \exists R$ s.t.

$(T-x) \cap B_R(0) = (T-y) B_R(0) \Rightarrow |f(x) - f(y)| < \varepsilon$

Now let Λ^k_{PE} = sPE k-forms and consider the cochain complex

$\Lambda^0_{PE} \xrightarrow{d} \Lambda^1_{PE} \xrightarrow{d} \Lambda^2_{PE}$ (d is the exterior derivative)

and define

$H^k_{PE} = \ker d_k / \operatorname{im} d_{k-1}$

Thm (Kellendonk, Putnam): $H^k_{PE}(T) = \check{H}(\Omega_T, \mathbb{R})$.

The only downside of this thm is that the coefficients of the cohom group are taken in $\mathbb{R}$, so we lose the info. that we would have had by taking coeffs. in $\mathbb{Z}$.

To recover this info we consider PE cochains. Let Λ^k_{PE} = PE cochains with coeffs. in A, and define H^k_{PE} accordingly.

Thm (Sadun): $H^k_{PE}(T) = \check{H}^k(\Omega_T, A)$.

Idea here: Any sPE cochain is a function which only depends on some finite radius R. Therefore it is actually a function which is uniquely defined by its values on some large enough approximant. The inverse limit structure implies that $H^k_{PE}(T)$ is the direct limit of $H^k_{PE}(\Gamma^n) = \check{H}^k(\Omega_T, A)$.
As a bonus, this construction allows us to realize generators of the cohomology of the tiling space by specifying sPE cochains coming from the inv. lim. structure.

Now let α by a sPE closed 1-cochain.

Lemma: Write $\alpha = \delta f$. Then f is wPE iff f is bounded. (Here we are assuming FLC and repetitivity.)
Pf. of one direction: Let $2M = \sup f - \inf f$, and suppose by translating if necessary that $\operatorname{im} f \subseteq [-M, M]$. Then $\exists$ p, q s.t. $f(p) - f(q) > 2M - 2\varepsilon$,

so $\int_q^p \alpha \geq 2M - 2\varepsilon$. Now using repetitivity we can find a radius R showing that f is wPE... $\square$

Referring to the thms discussed yesterday, we have

$$\frac{\text{Shape defs}}{\text{MLD}} = \frac{\{\alpha = \delta F \text{ sPE}\}}{\{\delta F \text{ sPE}\}} = \frac{\text{closed sPE 1-cochains}}{\text{exact sPE 1-cochains}}$$

$$= H^1_{PE}(T, \mathbb{R}^d)$$

Topological conjugacy $\Leftrightarrow$ F wPE, δF sPE, so

$$\frac{\text{Top conj}}{\text{MLD}} = \frac{\text{closed 1 cochain, sPE}}{\delta(\text{wPE}) \cap \text{sPE}} = H^1_{an}.$$

By the lemma, if we understand H^1_{an}, we understand the collection of 1 cochains whose integrals are bounded.

Question for next time: Consider $2 \to 1$ cut-and-project:

In these circumstances, we can view the function which counts points minus the expected number of points as a 1-cochain.

If this cochain is in H^1_{an} then the discrep. of points in the interval is bounded, and the converse is also true. This can be used to give a new proof of Kesten's thm.

Thm (Kesten, 1965; Kelly & Sadun, 2014): $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\delta \in (0,1)$, the quantity

$D_N = \#\{n \leq N : n\alpha \in (0,\delta) \bmod 1\} - N\delta$