Important facts which make Čech cohom.
the "right" invariant for us:
1) \( H^k(CW\, complex) = H^k(CW\, complex) \)
2) \( H^k(\lim (G^n, \rho_n)) = \lim H^k(G^n) \)

Review of direct limits:
Given a sequence of groups \( G^n \) and homoms \( \rho_n: G^n \to G^{n+1} \), the direct limit
\( \lim (G^n, \rho_n) \) is the disjoint union of the groups \( G^n \) modulo the equiv. relation
deﬁned by the rule \( x \sim \rho_n(x) \).

Ex: If \( G^n = \mathbb{Z} \) and \( \rho_n \) is multiplication by \( 2 \)
then \( \lim (G^n, \rho_n) \cong \mathbb{Z} [\frac{1}{2}] \) (the additive

group of rationals
with even denoms.)

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**Cut-and-project tilings**

\[ H = \mathbb{R}^d \times \Lambda \]

We deﬁne two projections
\[ \pi^n: H \times \mathbb{R}^d \to \mathbb{R}^d \quad \text{and} \quad \pi^t: H \times \mathbb{R}^d \to H, \] so that
\( \pi^n \) is injective.

Note: A viewpoint which is essentially equivalent is to take \( \Lambda = \mathbb{Z}^N \) and think of \( \mathbb{R}^d \) as meeting \( H \) with an "irrational" slope:
Sidenote: we are focusing on point patterns. However, if we take the Voronoi cells surrounding the points in such a pattern, and then take the dual tiling, we obtain a tiling whose vertices are exactly the points in the point pattern.

Now let \( \Phi \in \mathbb{R}^n \) we can define \( \chi_\Phi = \pi^{-1}((\Lambda + \Phi) \cap \mathbb{S}) \). We call \( \Phi \) nonsingular if none of the points of \( \Lambda + \Phi \) fall on \( \mathbb{S} \). Choose a nonsingular \( \Phi \) and consider \( \Omega_{\chi_\Phi} \).

There is a continuous map from \( \Omega_{\chi_\Phi} \) to \( \mathbb{H} \times \mathbb{R}^d / \Lambda \) given by sending a tiling \( \psi_\Phi \) to the parameter \( \Phi' \). Now there is an important issue that arises when we consider limit points. If we approach a singular point from different directions, we can get different limit points depending on whether or not the points were contained in the strip, in the limit. This means that topologically, the space of all tilings coming from our cut-and-project setup is a torus, with a set of measure 0 (the projection of \( \mathbb{S} \)) cut out and glued back in multiple times.

**Exercise:** Let \( n = 2, d = 1 \). The space of all tilings we can obtain looks like \( T^2 \) with one or two lines cut out (assuming \( \mathbb{W} \) is an interval). If the endpoints of \( \mathbb{W} \) are related by an element of \( \Lambda \) then there is one line cut out, otherwise there are two.
The resulting space can be realized as an inverse limit of punctured tori.

Since the maps in the limit give isomorphisms of cohomology, we can understand the topology of the resulting space by understanding the topology of the approximants.

The general description of topology of cut-and-project tilings is given in a book by Forrest, Hunton, and Kellendonk.

**PE cohomology**

\[ T = \text{tiling}, \quad f: \mathbb{R}^d \to A \text{ is strongly pattern equivariant (sPE) with radius } R \text{ if} \]
\[ (T-x) \cap B_K(0) = (T-y) \cap B_K(0) \Rightarrow f(x) = f(y) \]

- \( f \) is sPE if \( \forall R \) s.t.
- \( f \) is weakly PE if \( \forall \varepsilon > 0 \) \( \exists R \) s.t.

\[ (T-x) \cap B_K(0) = (T-y) \cap B_K(0) \Rightarrow |f(x) - f(y)| < \varepsilon \]

Now let \( \Lambda^k_{PE} = \text{sPE } k\text{-forms} \) and consider the cochain complex

\[ \Lambda^0_{PE} \xrightarrow{d} \Lambda^1_{PE} \xrightarrow{d} \Lambda^2_{PE} \quad (d \text{ is the exterior derivative}) \]

and define

\[ H^k_{PE} = \ker d_k / \text{im} d_{k-1} \]

**Thm (Kellendonk/Pulman):** \( H^k_{PE}(T) = \tilde{A}\left(\Lambda^k, \mathbb{R}\right) \),
The only downside of this theorem is that the coefficients of the cohomology group are taken in \( \mathbb{K} \), so we lose the info that we would have had by taking coefficients in \( \mathbb{Z} \).

To recover this info we consider PE cochains.

Let \( H^k_{PE} \) be PE cochains with coefficients in \( \mathbb{A} \), and define \( H^k_{PE} \) accordingly.

**Thm (Sadun)**: \( H^k_{PE}(\Omega_T) = \tilde{H}^k(\Omega_T, \mathbb{A}) \).

**Idea here**: Any PE cochain is a function which only depends on some finite radius \( R \).

Therefore, it is actually a function which is uniquely defined by its values on some large enough approximant. The inverse limit structure implies that \( H^k_{PE}(\Omega_T) \) is the direct limit of \( H^k_{PE}(R^n) = \tilde{H}^k(\Omega_{R^n}, \mathbb{A}) \).

As a bonus, this construction allows us to realize generators of the cohomology of the tiling space by specifying PE cochains coming from the map limit structure.

Now let \( \alpha \) be a PE closed 1-cochain.

**Lemma**: Write \( \alpha = sf \). Then \( f \) is w PE "iff" \( f \) is bounded. (Here we are assuming FLC and repetitivity.

**Proof of one direction**: Let \( 2M = \sup f - \inf f \), and suppose by translating if necessary that \( \text{im } f \subseteq [-M, M] \). Then \( \exists \ p, q \ s.t. \ f(p) - f(q) > 2M-2\varepsilon \)
so \( \int q \geq 2M - 2e \). Now using repetitivity we can find a radius \( R \) showing that \( f \) is wPE... 0

Returning to the things discussed yesterday, we have

\[
\text{shape def} = \begin{cases} \text{closed } \text{SF} \text{ sPE} & \text{exact } \text{SF} \text{ sPE} \\ \text{MLD} & \text{1-cochains} \end{cases}
\]

\[ = H'_{\text{PE}}(T, \mathcal{K^q}) \]

Topological conjugacy \( \Leftrightarrow \) F wPE, SF sPE, so

\[
\text{Top conj} = \begin{cases} \text{closed } 1 \text{ cochain } \text{sPE} & \text{H}^1 \text{in} \\ \text{MLD} & S(\text{wPE}) \text{nsPE} \end{cases}
\]

By the lemma, if we understand \( H^1 \), we understand the collection of 1 cochains whose integrals are banded.

Question for next time: Consider 2 \( \rightarrow \) 1 cut-and-project:

\[
\begin{array}{c}
\text{In these circumstances,} \\
\text{we can view the function which counts points minus} \\
\text{the expected number of} \\
\text{points as a 1-cochain.} \\
\text{If this cochain is in } H^1 \text{ then the discrepency of points in the interval is bounded, and the converse is also true. This can be used to give a new proof of Kesten's thm.}
\end{array}
\]
Thm (Kesten, 1965; Kelly, Sadun, 2014): $A \times \mathbb{Z} \setminus \mathbb{Q} \setminus \mathbb{G} / \langle 0,1 \rangle$, the quantity 

$D_N = \# \{ n \leq N : n \in (0,8) \mod 1 \} - N$