Topology of Tiling Spaces 4/4
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P.1/6

<u>How to compute cohomology for substitution tilings.</u>

First consider Thue-Morse: $a \mapsto ab$, $b \mapsto ba$

$\cdots \mid a \mid b \mid b \mid a \mid b \mid a \mid a \mid b \mid b \mid \cdots$

Instead of collaring tiles we will collar pts. Fix $\varepsilon > 0$ and define an equiv. rel. by $x \sim y \Leftrightarrow (T-x) \cap B_\varepsilon(0) = (T-y) \cap B_\varepsilon(0)$.
The <u>Barge-Diamond complex</u> is then defined to be $\Gamma_{BD} := T/\sim$. We identify

T with $\mathbb{R}$, which gives us the natural topology on Γ_{BD}. If $\varepsilon = 0$ then this gives the (uncollared) Anderson-Putnam complex. If ε is small and positive then we get

. The large pieces are called <u>tile cells</u>,

and the small pieces of length 2ε are called vertex flaps. Next we can form the BD-complex for supertiles of level 1 under the subst. map, with ε replaced by $\lambda\varepsilon$, where λ is the inflation factor. The new complex is homeomorphic to the original one and this allows us to represent Ω_T as an inv. lim: $\Gamma_{BD} \xleftarrow{\sigma} \Gamma_{BD} \xleftarrow{\sigma} \cdots$.

supertiles of lev. 1 w.r.t. $\lambda\varepsilon$

tiles w.r.t. ε

There is an issue, that is not exactly a simplicial map (e.g. maps edge flaps to something which overlaps with edge flaps and tile cells). However each σ is homotopy

P.2/6

equiv. to a simplicial map σ'. Therefore

$$\check{H}^k(\Omega_T) \cong \varinjlim \left(H^k(\Gamma_{BD}) \sigma^* \right)$$
$$= \varinjlim \left(H^k(\Gamma_{BD}), (\sigma')^* \right)$$
$$= H^k\left(\varprojlim (\Gamma_{BD}, \sigma') \right) =: \Omega'.$$

By choosing ε small enough we can ensure that σ' maps edge flaps to edge flaps. Then (working with a view towards using exact sequences / excision thms.) we let
$S_0 =$  , and $\Omega_0 = \varprojlim (S_0, \sigma') \subset \varprojlim (\Gamma, \sigma')$.

Then $H^1(S_0) = \mathbb{Z}$, and the map $(\sigma')^* : H^1(S_0) \to H^1(S_0)$ is multiplication by -1. Therefore $H^1(\Omega_0) = \mathbb{Z}$.

The homology of the vertex flaps tells us "how things in Ω_T are connected at ∞".

Next consider the action of σ' on Γ / S_0. This space is a wedge of two circles:  ,

so $H^1(\Gamma / S_0) \cong \mathbb{Z}^2$, and

$$H^1(\Omega / \Omega_0) = \varinjlim \left(H^1(\Gamma / S_0), M^t \right).$$
(substitution matrix)

Now we have an exact sequence

$$0 \to \mathbb{Z} \to H^0(\Omega_0) \to H^1(\Omega/\Omega_0) \to H^1(\Omega) \to H^1(\Omega_0) \to 0.$$

This gives

\# of tile types

$$0 \to \mathbb{Z}^{\#\{\text{con. comps}\}-1 \text{ of } \Omega_0} \to H^1(\mathbb{Z}^N, M^t) \to H^1(\Omega) \to \mathbb{Z}^{\#\text{loops}} \to 0$$

For Thue Morse we get

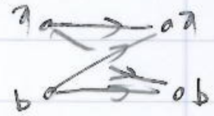
$$0 \to \varinjlim\left(\mathbb{Z}^2, \begin{pmatrix}1&1\\1&1\end{pmatrix}\right) \to H^1(\Omega) \to \mathbb{Z} \to 0$$

$$\Rightarrow H^1(\Omega) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}.$$

Next let's try the Fibonacci tiling, $a \mapsto ab$, $b \mapsto a$.

Γ_{BD} :  . Here S_0 is contractible, so $H^1(S_0) = 0$, and we obtain $H^1(\Omega) \cong \varinjlim\left(\mathbb{Z}^2, \begin{pmatrix}1&1\\1&0\end{pmatrix}\right) \cong \mathbb{Z}^2$.

One feature of the Fibonacci tiling is that, all supertiles of level 1 start with "a". Even if we started with the subst.

$a \mapsto abbaab$, $b \mapsto abba$, we would have begun with $S_0 =$  . However upon applying σ^1 we would have

$a \to b \;\to\; b \to a$

$a \to a \;\to\; b \to a$

$b \to b \;\to\; a \to a$

$b \to a \;\to\; a \to a$

This gives the smaller picture 

for supertiles, which results in $H^1(\Omega_0) = 0$.

What about for two dimensions? We consider all points in our tiling, together with their ε neighborhoods (w.r.t. the sup-norm metric, for convenience). Now in order to use the analogue of the technique used in 1 dim, we need information about H^0, H^1, and H^2 of all of the "tile cells" and "flaps". This requires more advanced algebraic topology, but it is possible.

Suppose that T_1 and T_2 are two tilings and that $\psi: \Omega_{T_1} \to \Omega_{T_2}$ is a topological conjugacy satisfying $\psi(T_1) = T_2$.

Def: An FLC Delone set Y is a <u>Meyer set</u> if $Y - Y$ is uniformly discrete.

Thm: If $\psi: \Omega_{T_1} \to \Omega_{T_2}$ is a top. conj. then $\psi = s \circ \phi$, where ϕ is an MLD map and s is a shape conjugacy.

Recall: ϕ is an MLD map if $\exists R_1, R_2$ s.t.

$$(T_1 - x) \cap B_{R_1}(0) = (T_1 - y) \cap B_{R_1}(0) \Rightarrow (\phi(T_1) - x) \cap B_1(0) = (\phi(T_1) - y) \cap B_1(0),$$

and $(\phi(T_1) - x) \cap B_{R_2}(0) = \cdots \Rightarrow \cdots = (T_1 - y) \cap B_1(0)$.

A map $Y \to Y'$ of point patterns is a shape conj. if it is given by $y \mapsto y + F(y)$, where F is wPE and δF is sPE.

If we want to understand what is preserved by top. conjugacies, it boils down to understanding what is preserved by shape conjugacies.

We want to construct a topological conjugacy which does not preserve the Meyer property. In order to do this consider the substitution

$a_1 \mapsto a_1 b_1 a_1$
$b_1 \mapsto a_1 b_2$
$a_2 \mapsto a_1 b_2 a_2$
$b_2 \mapsto a_2 b_1$

The expansion factor of the subst. is ϕ for a_1 and a_2 tiles, and 1 for b_1 and b_2 tiles. The subst. mat is $M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$.

The eigenvals. of the mat. are ϕ^2, ϕ^{-2}, ϕ, and $1-\phi$. The quantities $|\sigma^n(a_1)|$ and $|\sigma^n(a_2)|$ are both distances b/t elements of the corresp. pt. pattern, however $|\sigma^n(a_1)| - |\sigma^n(a_2)| \sim |-\phi|^{-n} \to 0$. Therefore it is not a Meyer set. However by erasing subscripts we see that this pt. pattern is top. conj. to the Fib. tiling.

For our next result we make two assumptions:

H1: Internal group $= C \times \mathbb{R}^{N-d}$, C finite

H2: Window is a finite disjoint union of polyhedra.

Thm: If Y' is top. conj to Y then Y' is MLD to a <u>reprojection</u> of Y (a projection onto the phys. sp. in a diff direction, i.e. a linear change of vars.)

Thm: All "coboundaries" are of the form reprojection + SPE.

Thm: $H^1_{an} = H^1_{as} = H^1_{repr.}$

In the example above, we had $\dim(H^1_{an}) = \dim(H^1_{repr})$ $= 1$, but $\dim(H^1_{an}) = 1$. This implies that the

window realizing that pattern as a cut and proj.
set is not a finite disj. union of polygons.

This idea is also what is used in the proof of
Kesten's thm by Kelly and Sadun.