Properties of topological groups and Haar measure

We will go through the basics of topological groups as well as give the definition and the basic theorem regarding the Haar measure. Most of the material comes from Chapter 1 of [1].

1 Topological Groups

A **topological group** is a group G equipped with a topology compatible with the group operations, i.e.

(TG1) The function $\phi : G \times G \to G$ defined by $\phi(g, h) = gh, \forall g, h \in G$, is continuous, where $G \times G$ has the product topology.

(TG2) The function $\phi : G \to G$ defined by $\phi(g) = g^{-1}, \forall g \in G$, is continuous.

Examples of topological groups are:$(\mathbb{R}^*, .)$ with the euclidean topology and $GL(n, \mathbb{R})$ under matrix multiplication, with the euclidean topology on $\mathbb{R}^{n\times n}$. A non-euclidean topological group is $(\mathbb{Q}_p, \|.\|_p)$ and in fact every field equipped with a norm can be turned into a topological group. By convention, whenever we refer to a finite topological group, we will always assume that it has the discrete topology.

Proposition 1.1. *(***Translation Invariance***) For any topological group $G, U \subseteq G$ and $g \in G$, the following are equivalent:*
(i) U is open,
(ii) gU is open,
(iii) Ug is open.
(iv) U^{-1} is open.

Proof. We will only prove the first implication as the others are similar. Let $g \in G$ be fixed and U be an open subset of G. We will show that gU is open by showing that for each of it's elements x, one can find an open set U_x containing x such that $U_x \subseteq gU$. Then the equality $gU = \cup_{x\in gU} U_x$ will prove that gU is open.

Now let $x \in gU$. One can therefore find $u \in U$ such that $x = gu$. Then the continuous function ϕ defined in (TG1) satisfies $\phi(g, u) = x$ and since gU is an open set which contains x, the definition of the product topology implies that one can find open sets U_1, U_2 containing g and u respectively, such that $U_1U_2 \subseteq gU$. We can then define $U_x := U_1U_2$ and notice that $x = gu \in U_1U_2 = U_x$ and that $U_x = \cup_{h \in U_2} U_1 h$ is open. $\square$

Definition 1.2. *(i)Let X be a topological space and $U \subseteq X, x \in X$. Then U is called a* **neighborhood** *of x if $x \in U^\circ$.*
(ii)A subset S of a group G is called **symmetric** *if it satisfies $S^{-1} = S$.*

The group theoretic structure of a topological group allows us to pick a basis consisted from nicer sets than the general open sets and this is shown in the first two sentences of the following Lemma. The rest of the sentences deal with some topological properties of subgroups of a topological group.

Lemma 1.3. *Let G be a topological group. Then*
(i) For each neighborhood U of the identity e, one can find a neighborhood V of e such that $VV \subseteq U$.
(ii) For each neighborhood U of the identity e, one can find a symmetric neighborhood V of e such that $V \subseteq U$.
(iii) Let H be a subgroup of G. Then $\overline{H}$ is also a subgroup of G.
(iv)An open subgroup of G is also closed.

Proof. (i) The restriction ϕ' of the function ϕ given in (TG1) to $U^\circ \times U^\circ$, gives rise to a continuous function whose inverse image of U° is a neighborhood of (e, e). One can therefore find neighborhoods V_1, V_2 of the identity, such that $V_1V_2 = \phi'(V_1 \times V_2) \subseteq U^\circ$. Letting $V := V_1 \cap V_2$ yields the result.
(ii)The set $V := U^\circ U^\circ$ is symmetric and a neighborhood of the identity due to Proposition 1.1(iv).
(iii) Assuming that $g, h \in \overline{H}$, one can find nets g_l, h_l in H, converging to g and h respectively.(TG2) implies that $h_l^{-1} \to h^{-1}$ and (TG1) implies that $g_l h_l^{-1} \to gh^{-1}$, i.e. $gh^{-1} \in \overline{H}$.
(iv)If H is a subgroup of G, then G is the disjoint union of cosets of H, one of which can be taken to be H itself. One therefore gets

$$G = (\cup_{g \notin H} gH) \cup H.$$

By translation invariance, each coset gH is open, and therefore the union of such sets is open as well. Then the previous equation shows that the compliment of H in G is open, which proves the assertion. $\square$

As already suggested, the topology of topological groups has special characteristics compared to any general topological space. The next proposition shows another instance of this phenomenon:

Proposition 1.4. *Let G be a topological group. Then the following are equivalent:*

(i) G is a T_1 topological space.
(ii) G is a T_2 topological space.
(iii) $\{e\}$ is closed.
(iv) $\{g\}$ is closed for any $g \in G$.

Proof. Translation invariance proves the implication (iii)⇒(iv). Both implications (ii)⇒(iii) and (iv)⇒(i) are standard facts in the context of general topology. To prove (i)⇒(ii) take any distinct group elements g, h. Now notice that the definition of a T_1 space implies the existence of a neighborhood U of the identity, such that $gh^{-1} \notin U$. By Lemma 2(i) and Lemma 2(ii), one can find a symmetric neighborhood V of the identity such that $VV \subseteq U$. Then the sets Vg, Vh are disjoint, otherwise $v_1 g = v_2 h$ for some $v_1, v_2 \in V$ and then $gh^{-1} = v_1^{-1} v_2 \in V^{-1}V = VV \subseteq U$ would be a contradiction. The two neighborhoods separate the points g and h and therefore our claim is proved. □

Definition 1.5. *Let H be a subgroup of a topological group G. The* **quotient topology** *on G/H is defined such that a set $U \subseteq G/H$ is open if and only if $\rho^{-1}(U)$ is open in the topology of G, where $\rho : G \to G/H$ is the canonical projection map.*

Notice that G/H is merely a topological space in general and not necessarily a topological group. The reason is H might not be a normal subgroup of G so G/H does not even have the structure of a group. However, as we shall later see, whenever the condition of normality is satisfied, then G/H forms a topological group.

Since the quotient topology is defined through a subgroup, one should expect that topological properties of $H \leq G$ should imply various properties for G/H of the same nature. In particular we have the next proposition:

Proposition 1.6. *Let G be a topological group and H be a subgroup of G. Then the following hold,*
(i) Each translation map on G/H is a continuous function.
(ii) The canonical projection $\rho : G \to G/H$ is an open map.
(iii) G/H is a T_1 topological space if and only if H is a closed subgroup of G.
(iv) G/H is a discrete topological space if and only if H is an open subgroup of G. If furthermore G is a compact topological group then G/H is a discrete finite topological space if and only if H is open.

Proof. (i)This is an immediate corollary of the translation invariance of the group G and the definition of the quotient topology.
(ii)Let V be an open subset of G. Then Definion 5 implies that the image $\rho(V)$ is open in G/H if and only if $\rho^{-1}(\rho(V))$ is open in G. This set however is easily seen to equal VH, which is the union of the translates Vh as h ranges through H and is therefore an open set.
(iii) The topological space G/H is T_1 if and only if each of its singletons are closed, which is a classical fact regarding T_1 spaces. However, part (i) of this proposition implies that this is holds if and only if the singleton $\{H\}$ is closed. By the definition of the quotient topology, this is equivalent to $\rho^{-1}(\{H\}) = H$

being closed in G.
(iv) G/H is a discrete topological space if and only if each of its singletons $\{gH\}, \forall g \in G$, is open. As in the proof of (iii) this is equivalent to H being open in G. If G is a compact topological space, then it's continuous image $\rho(G) = G/H$ is a compact topological space as well. Since any space is the union of its singletons, the definition of compactness shows that discreteness implies finitiness. $\square$

Remarks:(1) Proposition 1.6(ii) implies that G/H is a topological group whenever H is a normal subgroup of G.
(2) One can show that every topological group projects by a continuous homeomorphism to a T_2 topological group, and therefore the extra supposition that a topological group is Hausdorff is not particularly demanding. To prove this, let $H := \overline{\{e\}}$, so that H is a subgroup of G. In fact it is a normal subgroup : for each $g \in G$, the set $g^{-1}Hg$ is closed and contains the identity, hence by the definition of the topological closure, one has $H \subseteq g^{-1}Hg$, i.e. $gHg^{-1} \subseteq H$. Therefore the topological space G/H is a topological group by the previous remark. Since H is closed, Proposition 1.6(iii) implies that G/H is T_1 which by Proposition 1.4 shows that G/H is a Hausdorff topological group.

The following proposition follows the same spirit of converting tpological information about $H \leq G$ to topological information about G/H. Notice that the proof procees uses net limits and therefore the Hausdorff condition is needed to guarantee uniqueness of limits.

Proposition 1.7. *Let G be a Hausdorff topological group. If H is a compact subgroup, then the canonical projection $\rho : G \to G/H$ is a closed map.*

Proof. Let $X \subseteq G$ be closed. Then $\rho(X)$ is closed if and only if XH is closed in G, as argued in the proof of Proposition 1.6(ii). We will prove this by using nets : If $z \in \overline{XH}$ then it is the limit of a net $x_l h_l$, where $x_l \in X, h_l \in H$. The compactness of H implies that one can find a subnet of h_l which converges to a limit $h \in H$ and we henceforth focus on this subnet, identifying it with the original net. By (TG1) and (TG2), one gets

$$x_l = (x_l h_l) h_l^{-1} \to zh^{-1},$$

and therefore $zh^{-1} \in X$, since X is closed. This implies that $z = xh \in XH$ and hence $\overline{XH} = XH$. $\square$

Measures on topological groups will be introduced later, but their existence can only be guaranteed in general only for the following kind of topological groups:

Definition 1.8. *A* **locally compact topological group** *is a Hausdorff topological group for which each point has a compact neighborhood.*

Proposition 1.9. *Let G be a Hausdorff topological group. Then any locally compact subgroup is closed. In particular, each discrete subgroup of G is closed.*

The proof can be found in [1].

2 Haar Measure

In this section we briefly introduce the notion of Haar measure and give a few examples.

Definition 2.1. *A* **Radon measure** *is a Borel measure on a Hausdorff locally compact topological space which is finite on compact sets, inner and outer–regular on all open sets.*

Definition 2.2. *A* **Haar measure** *on a locally compact topological group G is a non–zero Radon measure which is right translation –invariant, i.e.*

$$\mu(gE) = \mu(E)$$

for any Borel subset E of G and each $g \in G$.

One can similarly define left translation–invariant measures, or bi-invariant translation –invariant measures, which is a combination of both.

The existence part of the following theorem was proved by Haar [2] and the uniqueness part by Weil [3].

Proposition 2.3. *Each locally compact topological group G admits a Haar measure, which is unique up to scalars.*

A first example of Haar measures is by $G = GL(n, \mathbb{R})$, equipped with matrix multiplication and the euclidean topology, where the measure is defined as

$$\mu(S) := \int_S \frac{dX}{|\det(X)|}.$$

A second example is the group $SL_2(\mathbb{R}) \simeq \mathbb{H} \times \mathbb{C}$, where the measure is given through the hyperbolic metric.

The next example we will consider, is given by the locally compact additive topological group $\mathbb{Z}_p$. By Proposition 2.3, we have an additive Haar measure and we normalise it so as to have $\mu(\mathbb{Z}_p) = 1$. Note that this can be done because $\mathbb{Z}_p$ is compact. For any $m \geq 1$, one may use the p–adic expansion in $\mathbb{Z}_p$ to show that the subgroup $p^m\mathbb{Z}_p$ has p^m representatives, from which we infer $\mu(p^m\mathbb{Z}_p) = p^{-m}$.

Now let us compute

$$I_p := \int_{\mathbb{Z}_p} \|x\|_p \; d\mu.$$

On noticing that $W_i := \{x \in \mathbb{Z}_p : v_p(x) = i\} = p^i\mathbb{Z}_p - p^{i+1}\mathbb{Z}_p$, we get

$$I_p = \sum_{i \geq 0} p^{-i}\mu(W_i) = \sum_{i \geq 0}(p^{-2i} - p^{-2i-1}) = p(p+1)^{-1}.$$

References

[1] D. Ramakrishna, R. Valenza, *Fourier Analysis on Number Fields*, Springer Verlag, New York, 1999.

[2] Haar, A. (1933), Der Massbegriff in der Theorie der kontinuierlichen Gruppen, *Annals of Mathematics*, 2 34 (1), 147-169.

[3] Weil, A., *L'integration dans les groupes topologiques et ses applications*, Actualits Scientifiques et Industrielles, 869, Paris, Hermann, 1940.