Adeles and Ideles

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1 p-adic solenoids

Consider the group $\mathbb{R} \times \mathbb{Q}_p$. This is clearly a topological group, where the topology is the product topology, and it has some nice properties.

First of all, it is locally compact, since $\mathbb{R}$ and $\mathbb{Q}_p$ are. (If $x \in \mathbb{R}$ has compact neighbourhood N_x, and $y \in \mathbb{Q}_p$ has compact neighbourhood N_y, then $N_x \times N_y$ is a compact neighbourhood of $(x, y) \in \mathbb{R} \times \mathbb{Q}_p$.)

Secondly, it has a discrete subgroup $\mathbb{Z}[\frac{1}{p}]$ which is *cocompact*: that is, the quotient group $(\mathbb{R} \times \mathbb{Q}_p)/\mathbb{Z}[\frac{1}{p}]$ is a compact group. (To see this, we can show that $[0, 1] \times \mathbb{Z}_p$ is a fundamental domain: if $(x, y) \in \mathbb{R} \times \mathbb{Q}_p$, then we have

$$\underbrace{\frac{a_{-n}}{p^n} + \frac{a_{-(n-1)}}{p^{n-1}} + \cdots + \frac{a_{-1}}{p}}_{\in \mathbb{Z}[\frac{1}{p}]} + \underbrace{a_0 + a_1 p + a_2 p^2 + \cdots}_{\in \mathbb{Z}_p}.$$

So subtract off the bit not in $\mathbb{Z}_p$ from both x and y to get x' and y': then we have $x' \in \mathbb{R}$ and $y' \in \mathbb{Z}_p$. Then subtract off $\lfloor x' \rfloor$ from both x' and y' to get $x'' \in [0, 1)$ and $y'' \in \mathbb{Z}_p$.) This quotient is called the p-adic solenoid.

If we now consider taking $\mathbb{R} \times \mathbb{Q}_{p_1} \times \mathbb{Q}_{p_2}$, or even $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_n}$, this is still fine: we have a topological group, with topology the product topology, which is locally compact and has a discrete cocompact subgroup $\mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_n}]$, with quotient like $[0, 1] \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_n}$.

2 Problems with extending this

Now, given we have the direct product of $\mathbb{R}$ with any finite number of $\mathbb{Q}_{p_i}$ and that all works fine, it's natural to ask whether we could in fact take the direct product of $\mathbb{R}$ with *all* the p-adic fields: that is, take

$$\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \cdots .$$

However, in this case, we run into several problems.

Firstly, an infinite product of locally compact groups equipped with the product topology is not necessarily locally compact: since any open set in the product is the product of open sets from the individual spaces forming the product, where all but finitely many of these open sets must be the whole space.

Secondly, we can't find a cocompact discrete subgroup of the space.

How do we remedy this?

3 Restricted direct products

Definition. *Let $J = \{v\}$ be a set of indices, and J_∞ a fixed finite subset of J. Suppose that for every index v we are given a locally compact group G_v, and for all $v \notin J_\infty$, we are also given a compact open subgroup H_v of G_v.*

Then we define the RESTRICTED DIRECT PRODUCT *of the G_v with respect to the H_v as*

$$\prod_{v\in J}' G_v = \{(x_v) \mid x_v \in G_v \text{ with } x_v \in H_v \text{ for all but finitely many } v\}.$$

The topology on this restricted direct product is given by specifying a neighbourhood base of the identity 1, consisting of all sets of the form $\prod N_v$, where N_v is a neighbourhood of 1 in G_v and $N_v = H_v$ for all but finitely many v.

Later, we're going to use this to consider $G_\infty = \mathbb{R}$ and $G_p = \mathbb{Q}_p$ with $H_p = \mathbb{Z}_p$; for now, we're just going to derive some general properties of restricted direct products.

Proposition. *Let G be the restricted direct product of G_v with respect to H_v. Then G is a locally compact group.*

Proof. Let S be any finite subset of J containing J_∞: consider

$$G_S = \prod_{v\in S} G_v \times \prod_{v\notin S} H_v.$$

Then G_S is locally compact in the product topology, using Tychonoff's theorem and the fact that all of the H_v are compact. But the product topology on G_S is identicaly to that induced by our original topology on G. Hence every subgroup of the form G_S is locally compact with respect to the topology of the restricted direct product; since every $x \in G$ belongs to some subgroup of this form, it follows that G is locally compact. End of proof.

We can also extend Haar measure onto restricted direct products: we will simply state this proposition without proof.

Proposition. *Let*

$$G = \prod_{v\in J}' G_v$$

be the restricted direct product of the groups G_v wrt the subgroups H_v. Let dg_v denote the corresponding Haar measure on G_v, normalised so that

$$\int_{H_v} \mathrm{d}g_v = 1$$

for almost all $v \notin J_\infty$. Then there is a unique Haar measure dg on G such that for each finite set of indices S containing J_∞, the restriction dg_S of dg to

$$G_S = \prod_{v \in S} G_v \times \prod_{v \notin S} H_v$$

is precisely the product measure.

4 The adeles and approximation

Now, we consider a specific example. Let $J = \{\infty, 2, 3, 5, \ldots\}$, with $J_\infty = \{\infty\}$, and let $G_\infty = \mathbb{R}$, with $G_p = \mathbb{Q}_p$ and $H_p = \mathbb{Z}_p$. Then the restricted direct product of the $\mathbb{Q}_v$ is called the ADELE GROUP of $\mathbb{Q}$, and denoted $\mathbb{A}_\mathbb{Q}$.

Theorem.(Approximation Theorem for $\mathbb{Q}$) *We have*

$$\mathbb{A}_\mathbb{Q} = \mathbb{Q} + \left(\mathbb{R} \times \prod_p \mathbb{Z}_p \right).$$

Moreover, $\mathbb{Q} \cap \mathbb{A}_\infty = \mathbb{Z}$.

Proof. We need to show that given $x \in \mathbb{A}_\mathbb{Q}$, there exists some $\alpha \in \mathbb{Q}$ such that each component of $x - \alpha$ is a local integer. All but a finite number of x's components are already integers: so there exists some $m \in \mathbb{N}$ such that mx is integral at all finite primes. Let

$$m = p_1^{a_1} \cdots p_r^{a_r}.$$

By Chinese Remainder Theorem, find λ such that

$$mx_i \equiv \lambda \mod p_i^{n_i}.$$

Then let $\alpha = \frac{\lambda}{m}$. End of proof.

Theorem. $\mathbb{Q}$ *is a discrete, cocompact subgroup of* $\mathbb{A}_\mathbb{Q}$.

5 Ideles

The ideles are similar to the adeles, only rather than using additive groups, they use multiplicative groups.