Pontryagin Duality

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For the reading group, Topological Groups

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We follow [1] closely.

Let $G$ be an abelian topological group. The Pontryagin dual of $G$ is the group of continuous homomorphisms $G \to S^1 \subset \mathbb{C}$, and we now describe its (compact-open) topology. A neighbourhood base of the trivial character is given by the set of all

$$W(K, V) = \{ \chi \in \hat{G} : \chi(K) \subseteq V \},$$

for compact $K \subseteq G$ and $V \subseteq S^1$ a neighbourhood of 1. The set of all $W(K, V)$ and their translates therefore forms a base for the topology on $\hat{G}$.

### Proposition 1

Let $N = \{ e^{2\pi i \theta} : -\frac{1}{3} < \theta < \frac{1}{3} \}$. Then

(i) A group homomorphism $\chi : G \to S^1$ is continuous if and only if $\chi^{-1}(N)$ is a neighbourhood of the identity in $G$.

(ii) The set of $W(K, N)$, for compact $K \subseteq G$, is a neighbourhood base for the trivial character.

(iii) If $G$ is discrete then $\hat{G}$ is compact.

(iv) If $G$ is compact then $\hat{G}$ is discrete.

(v) If $G$ is locally compact then $\hat{G}$ is locally compact.

### Proof

(i) We don’t need arbitrarily small neighbourhoods $V$, for we can just take sufficiently large compact $K \subseteq G$. The technical details are in [1].

(ii) Likewise.

(iii) Assume that $G$ is discrete. Then $\hat{G}$ is the set of group homomorphisms $G \to S^1$. The topology of pointwise convergence on $\hat{G}$ is inherited from

\footnote{If we allow $V$ to be any open subset of $S^1$ then the $W(K, V)$ form a base for the topology. Hence this is the compact-open topology (in general these will only form a subbase).}

\footnote{Recall that a topological group is locally compact if it is locally compact and Hausdorff as a topological space.}
the product topology on \((S^1)^G = \{ f : G \to S^1 \}\). The projection maps are, for \(g \in G\),

\[
p_g : (S^1)^G \to S^1 \\
    f \mapsto f(g).
\]

A base for the product topology is given by finite intersections of \(p_g^{-1}(U)\), with \(g \in G\) and \(U\) open in \(S^1\). As the compact subsets of \(G\) are precisely the finite ones (since \(G\) is compact), the compact-open topology on \(\hat{G}\) matches the topology of pointwise convergence.

By Tychonoff’s theorem, \((S^1)^G\) is compact, so it remains to show that \(\hat{G}\) is a closed subset of \((S^1)^G\). Let \(f : G \to S^1\) be a limit point of \(\hat{G}\). Suppose for the sake of contradiction that \(f\) is not a homomorphism. Then there exist \(g, h \in G\) such that \(|f(gh) - f(g)f(h)| = 3\varepsilon\) for some \(\varepsilon > 0\). Every neighbourhood of \(f\) contains a homomorphism, and in particular there exists a homomorphism \(F : G \to S^1\) such that

\[
\begin{align*}
    |f(g) - F(g)| &< \varepsilon, \\
    |f(h) - F(h)| &< \varepsilon, \text{ and} \\
    |f(gh) - F(gh)| &< \varepsilon.
\end{align*}
\]

As \(F(gh) = F(g)F(h)\) and \(|F(g)| = |f(h)| = 1\), we now have

\[
3\varepsilon = |f(gh) - f(g)f(h)|
\]

\[
\leq |f(gh) - F(gh)| + |F(g)F(h) - F(g)f(h)| + |F(g)f(h) - f(g)f(h)|
\]

\[
< 3\varepsilon,
\]

contradiction. We conclude that \(\hat{G}\) is closed in \((S^1)^G\), and therefore compact.

(iv) Assume that \(G\) is compact. Then the subset \(W(G, N) = \{\chi_0\}\) is open in \(\hat{G}\), where \(\chi_0 : G \to S^1\) is the trivial character \(g \mapsto 1\). Its translates are therefore also open, so \(\hat{G}\) is discrete.

(v) See [1].
Theorem 2. Let $G$ be a locally compact abelian group (LCA). Then the evaluation map

$$
\alpha : G \to \hat{G}
$$

$$
g \mapsto \alpha(g) : \hat{G} \to S^1
$$

$$
\chi \mapsto \chi(g)
$$

is an isomorphism of topological groups.

Our focus will be the proof of this theorem. We can put Haar measure on $G$ since it’s LCA. We’ll take the theory of positive definite functions as a black box. Fourier inversion holds pointwise for continuous, $L^1$, positive definite functions to $G \to \mathbb{C}$. Let $V^1(G)$ denote the set of such functions.

For $f \in L^1(G)$, define

$$
L_zf : G \to S^1
$$

$$
t \mapsto f(z^{-1}t).
$$

Lemma 3. The map $\alpha$ is injective.

Proof. Let $z \in G\setminus\{1\}$. Suppose, for the sake of contradiction, that $\alpha(z) = 1$. Then $\chi(z) = 1$ for all $\chi \in \hat{G}$. For $f \in L^1(G)$ and $\chi \in \hat{G}$, using the definition of Haar measure,

$$
\hat{L_zf}(\chi) = \int_G f(z^{-1}y)\overline{\chi(y)}dy = \int_G f(y)\overline{\chi(y)}dy = \hat{f}(\chi).
$$

Hence $\hat{f} = \hat{L_zf}$ for all $f \in L^1(G)$. By Fourier inversion, $L_zf = f$ for all $f \in V^1(G)$.

As $G$ is Hausdorff\[3\] there exists an open neighbourhood $U$ of the identity such that $U \cap (z^{-1}U) = \emptyset$, and we may choose $U$ small enough to lie within a compact neighbourhood of the identity. Using Urysohn’s lemma, we can show that there exists a continuous, positive definite function $f \neq 0$ with support in $U$. Then $f \in L^1(G)$, being compactly supported, and so $f \in V^1(G)$. Now the supports of $f$ and $L_zf$ are disjoint, contradicting $L_zf = f$. \[3\]

\[3\]Recall that this is part of the definition of a locally compact topological group.
For a compact neighbourhood $\hat{K}$ of the identity character in $\hat{G}$ and an open
neighbourhood $V$ of the identity in $S^1$, let

$$W(\hat{K}, V) = \{ \psi \in \hat{\hat{G}} : \psi(\hat{K}) \subseteq V \}.$$ 

These subsets and their translates form a base for the topology of $\hat{\hat{G}}$.
We use these to construct a base for the topology of $G$. Put

$$W_G(\hat{K}, V) = W(\hat{K}, V) \cap \alpha(G),$$
and regard these as subsets of $G$ (since $\alpha$ is injective).

**Proposition 4.** The subsets $W_G(\hat{K}, V)$ and their translates form a base for
the topology of $G$.

*Proof. (sketch)* Let $U$ be an open neighbourhood of the identity $e \in G$. Use
Urysohn’s lemma to construct a continuous, positive definite function $g : G \to \mathbb{C}$
with support contained in $U$ such that $g(e) = 1$. Use the Fourier
transform and some measure theory to show that $g \simeq 1$ on $W_G(\hat{K}, V)$ (for
large enough $\hat{K}$ and small enough $V$), thereby establishing that

$$W_G(\hat{K}, V) \subseteq \text{supp}(g) \subseteq U. \quad (8)$$

\[\square\]

**Corollary 5.** The map $\alpha$ is bicontinuous (open and continuous), so $\alpha$ is a
homeomorphism onto its image.

*Proof. By construction*\[\square\]

$$\alpha(W_G(\hat{K}, V)) = W(\hat{K}, V) \cap \alpha(G). \quad (9)$$

This shows that $\alpha$ is bicontinuous at the identity, and the result follows by
translation. \[\square\]

**Corollary 6.** The image of $\alpha$ is closed in $\hat{\hat{G}}$.

\[4\text{Recall that the } W_G(\hat{K}, V) \text{ are considered as subsets of } G.\]
Proof. Equivalently, we show that $\alpha(G)$ is closed in its closure. Since every open subgroup of a topological group is closed it suffices to show that $\alpha(G)$ is open in its closure. This follows because $\alpha(G)$ is locally compact and dense in its closure. \hfill \Box

It remains to show that $\alpha(G)$ is dense in $\hat{G}$. Refer to the book if you’re interested.

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5Efthymios covered this: write the whole group as a disjoint union of cosets. Efthymios also showed us that the closure of a subgroup is a subgroup.

6Locally compact and dense in a Hausdorff space implies open.
References