Essential Extract: Vector Calculus

$**$ For videos see Advanced Math Cafe on YouTube $**$

Line integrals (three kinds)

$$f : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{F} : \mathbb{R}^n \to V_n$$

$$C: \ \mathbf{r}(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle, \ a \leqslant t \leqslant b$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

$$\textcircled{1} \ \int_C f \ ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \ dt$$

$$\textcircled{2} \ \int_C f \ dx_i = \int_a^b f(\mathbf{r}(t)) x_i'(t) \ dt$$

$$\textcircled{3} \ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

grad, div, and curl

$$\text{grad}(f) = \nabla f = \langle f_x, f_y, f_z \rangle$$

$$\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$$

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Conservativity, independence of path

$\mathbf{F} : D \to V_3$, $D \subseteq \mathbb{R}^3$ simply connected, first partials of $\mathbf{F}$ continuous:

$\mathbf{F} = \nabla f$ for some $f : D \to \mathbb{R}$

$\Leftrightarrow \int_C \mathbf{F} \cdot d\mathbf{r}$ independent of path

$\Leftrightarrow$ curl $\mathbf{F} = \mathbf{0}$

Surface integrals (two kinds)

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad \mathbf{F} : \mathbb{R}^3 \to V_3$$

$$S: \ \mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \ (u,v) \in D \subseteq \mathbb{R}^2$$

$$\mathbf{n}(u,v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

$$\textcircled{1} \ \iint_S f \ dS = \iint_D f(\mathbf{r}(u,v)) |\mathbf{r}_u \times \mathbf{r}_v| \ dA$$

$$\textcircled{2} \ \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \ dS = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \ dA$$

FTC for line integrals : $\displaystyle\int_C (\nabla f) \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$

Green's theorem : $\displaystyle\oint_{\partial D} P(x,y) \ dx + Q(x,y) \ dy = \iint_D (Q_x - P_y) \ dA$

Stokes's theorem : $\displaystyle\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \oiint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$

Divergence theorem : $\displaystyle\oiint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} \ dV$