1. (1 point) Prove that $\mathbb{Z}_n$ is a field if and only if $n$ is a prime number.

2. (1 point) Let $R$ be a ring with unity. Assume that for all $x$ and $y$ in $R$ we have $(xy)^2 = x^2y^2$. Prove that $R$ is commutative.

3. Let $S = \{q \in \mathbb{Q} : q = \frac{a}{b}, a, b \in \mathbb{Z}$ and $b$ odd\}.
   
   (a) (0.5 points) Prove that $S$ is a subring of $\mathbb{Q}$.
   (b) (0.5 points) Prove that $S$ has a unique maximal ideal.

4. Let $R$ be a commutative ring with unity $1 \neq 0$.
   
   (a) (0.5 points) Prove that $R$ is an integral domain if and only if $\{0\}$ is a prime ideal in $R$.
   (b) (0.5 points) Prove that $R$ is a field if and only if $\{0\}$ is a maximal ideal in $R$.

5. (2 points) Let $I$ be an ideal in the commutative ring $R$. Define 
   \[
   \text{rad}(I) = \{r \in R \mid \exists n \in \mathbb{N} : r^n \in I\}.
   \]
   Prove that $\text{rad}(I)$ is an ideal with $I \subset \text{rad}(I)$.

6. (2 points) Let $R$ be a commutative ring and $I$ a prime ideal. Prove that $\text{rad}(I) = I$.

7. Which of the following is a ring homomorphism? Prove your answer.
   (a) (1 point) $\varphi : \mathbb{R} \to \mathbb{R}, \varphi(x) = |x|$, 
   (b) (1 point) $\varphi : \mathbb{C} \to \mathbb{C}, \varphi(a + ib) = a - ib$.  