1. (1 point) Let $G = \{ e, x_1, \ldots, x_{r-1} \}$ be an abelian group such that $r = \#G$ is an odd integer. Prove that $x_1 \cdot \ldots \cdot x_{r-1} = e$.

Hint: Prove first that $x_1 \cdot \ldots \cdot x_{r-1}$ is its own inverse. Carefully explain your reasoning.

2. (a) (1 point) Find the right cosets of the subgroup $H = \{ (0, 0), (1, 0), (2, 0) \}$ in $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(b) (1 point) Find the right cosets of the subgroup $H = \{ (0, 0), (0, 2) \}$ in $\mathbb{Z}_3 \times \mathbb{Z}_4$.

3. (2 points) Prove that $\mathbb{Q}, +) / (\mathbb{Z}, +)$ is an infinite group such that each of its elements has finite order.

4. (1 point) Let $G$ be a group and let $H, K$ be two normal subgroups of $G$ with $H \cap K = \{ e \}$. Prove that for $x \in H$ and $y \in K$, $xy = yx$ holds.

5. (2 points) Let $G$ be a group and let $N$ a normal subgroup of $G$. Let $H$ be a subgroup of $G$. Set $NH = \{ nh \mid n \in N, h \in H \}$. Prove that $NH$ is a subgroup of $G$.

6. A subgroup $H$ of a group $G$ is characteristic if $\varphi(H) \subseteq H$ for every automorphism $\varphi$ of $G$.

(a) (1 point) Prove that every characteristic subgroup is normal.

(b) (1 point) Prove that the converse of (a) is false.