(1) Note that a finite integral domain is a field. Then it's sufficient to show that $\mathbb{Z}_n$ has no zero-divisors iff $n$ is prime.

$\mathbb{Z}_n$ has no zero-divisors if and only if for $[a], [b] \in \mathbb{Z}_n$ we have $[ab] = [0] \iff [a] = [0]$ or $[b] = [0]$. Then we have $n|ab \iff n|a$ or $n|b$ for $a, b \in \mathbb{Z}$. This holds only when $n$ is prime.

(2) From assumption we have $x(xy - yx)y = 0$. The problem becomes very easy if $R$ has no zero divisor. However, if $R$ might has zero divisor we may not naively cancel $x$ and $y$. We need some clever way to do this.

Use the assumption on $(1-x)y$ we have

$$(1 + x)y(1 + x)y = (1 - x)^2y^2$$

Simplify we get $yxy = xy^2$, or $(xy - yx)y = 0$. Then substitute $y$ with $(y + 1)$:

$$(y + 1)x(y + 1) = x(y + 1)^2$$

Simplify we get $yx = xy$.

(3) $\Rightarrow$ If $a$ is a unit then every $r \in R$ we have $r = r1 = raa^{-1} = a(ra^{-1}) \in aR$.

$\Leftarrow$ If $aR = R$ then $1 \in aR$. Then there exists $b \in R$ s.t. $1 = ab$.

(4) $a\frac{b_1}{b_2}, a\frac{b_2}{b_1} \in S$, then $\frac{a_1}{b_1} +\frac{a_2}{b_2} = \frac{a_1b_2 + a_2b_1}{b_1b_2}$ where $b_1, b_2$ are odds so $b_1b_2$ is odd, then $\frac{a_1}{b_1} +\frac{a_2}{b_2} \in S$.

Claim: $S_0 = \{ a \in S : a \text{ is even} \}$ is the unique maximal ideal. Actually, every element in $S - S_0$ is unit. For $\frac{a}{b} \in S - S_0$, $a$ is odd, so its inverse $\frac{a}{b} \in S$, then it is unit.

(5) Note that $\{0\}$ is the kernel of the identity map $id : R \to R$. The rest of proof follows from the fact that $R/I$ is integral domain iff $I$ is prime and is a field iff $I$ is maximal.