UH - Math 4377/6308 - Dr. Heier - Fall 2010

Midterm Exam

Wednesday, October 20, 2010

Print your NAME: Solution

Solve all of the four problems. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit. This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. The time allowed will be announced by the proctor.

Problem 1 ________	/20 points
Problem 2 ________	/30 points
Problem 3 ________	/25 points
Problem 4 ________	/25 points
Total ________	/100 points

1a. (10 points) Solve $z^2 - 4z + 5 = 0$ in $\mathbb{C}$ (i.e., the field of complex numbers).

1b. (5 points) Let $Z : \mathbb{R}^2 \to \mathbb{R}^2, (a_1, a_2) \mapsto (0,0)$ be the zero transformation. Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T \neq Z$, but $T \circ T = Z$. Show explicitly that your example has the required properties.

1c. (5 points) Let $Z : \mathbb{R}^3 \to \mathbb{R}^3, (a_1, a_2, a_3) \mapsto (0,0,0)$ be the zero transformation. Give an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T \circ T \neq Z$, but $T \circ T \circ T = Z$. Show explicitly that your example has the required properties.

1a) $az^2 + bz + c = 0 \iff z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\rightsquigarrow z = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm \frac{1}{2} \cdot \sqrt{-4} = 2 \pm \frac{1}{2} \cdot (2i) = \underline{\underline{2 \pm i}}$$

1b) $T(a_1, a_2) = (0, a_1)$ ($\neq Z$ obviously)

$T(T(a_1, a_2)) = T(0, a_1) = (0,0)$.

1c) $T(a_1, a_2, a_3) = (0, a_1, a_2)$

$T(T(T(a_1, a_2, a_3))) = T(T(0, a_1, a_2))$

$= T(0, 0, a_1) = (0,0,0)$

2a. (10 points) Find the condition on a, b, c so that

$$(a, b, c) \in \operatorname{span}\{(2,4,0), (3,3,3), (2,1,3)\}.$$

2b. (20 points) Find bases for the null space and range of

$T : \mathbb{R}^4 \to \mathbb{R}^3, (a_1, a_2, a_3, a_4) \mapsto (2a_1 - a_2 - a_3 - 2a_4, -a_1 + a_3 + a_4, 5a_1 - 3a_2 - 2a_3 - 5a_4).$

2a) $a_1 \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + a_3 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\Leftrightarrow \begin{aligned} 2a_1 + 3a_2 + 2a_3 &= a \\ 4a_1 + 3a_2 + a_3 &= b \\ 3a_2 + 3a_3 &= c \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} 2a_1 + 3a_2 + 2a_3 &= a \\ -3a_2 - 3a_3 &= b - 2a \\ 3a_2 + 3a_3 &= c \end{aligned}$$

$\Leftrightarrow$ Eqn 1

Eqn 2

$\underline{0 = c + b - 2a}$

The condition is $\underline{c = 2a - b}$.

2b) Find basis for null space first.

$$\begin{aligned} 2a_1 - a_2 - a_3 - 2a_4 &= 0 \\ -a_1 + a_3 + a_4 &= 0 \\ 5a_1 - 3a_2 - 2a_3 - 5a_4 &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} -a_1 + a_3 + a_4 &= 0 \\ -a_2 + a_3 &= 0 \\ \cancel{3a_2 + 3a_3} &\cancel{= 0} \end{aligned}$$

$\Leftrightarrow a_1 = a_3 + a_4$ and $a_2 = a_3$

$\leadsto N(T) = \left\{ \begin{pmatrix} a_3 + a_4 \\ a_3 \\ a_3 \\ a_4 \end{pmatrix} \,\middle|\, a_3, a_4 \in \mathbb{R} \right\}$

$= \left\{ a_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, a_3, a_4 \in \mathbb{R} \right\}$

$\Rightarrow$ basis is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

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Dim-formula:

$$\text{rank}(T) = 4 - \text{nullity} = 4 - 2 = 2.$$

$\leadsto$ Need to find 2 linearly indep. vectors in the Range(T) to get the basis.

$$T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}2\\-1\\5\end{pmatrix} \checkmark$$

$$T\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\-3\end{pmatrix}$$ not a multiple of the 1st, so also OK.

$\leadsto$ $\left\{ \begin{pmatrix}2\\-1\\5\end{pmatrix}, \begin{pmatrix}-1\\0\\3\end{pmatrix} \right\}$ is a basis of Range(T).

3a. (15 points) Let $T : \text{Mat}_{2\times2}(\mathbb{R}) \to \text{Mat}_{2\times2}(\mathbb{R}), T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ d & 3a \end{pmatrix}$. Let $\beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$. Compute $[T]_\beta^\beta$.

3b. (10 points) Let $\{v_1, v_2, v_3\}$ be a basis for $\mathbb{R}^3$. Is $\{v_1 + v_2, v_2 + v_3, v_1 - v_3\}$ also a basis for $\mathbb{R}^3$? Prove your answer.

3a) $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 3\cdot\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\cdot\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\to [T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{pmatrix}$

3b) $a(v_1+v_2) + b(v_2+v_3) + c(v_1 - v_3) = \vec{0}$

$\Leftrightarrow (a+c)v_1 + (a+b)v_2 + (b-c)v_3 = 0$

$\underset{v_1, v_2, v_3 \text{ lin. indep.}}{\Leftrightarrow}$ $\begin{aligned} a + c &= 0 \\ a + b &= 0 \\ b - c &= 0 \end{aligned}$ $\Leftrightarrow$ $\begin{aligned} a + c &= 0 \\ b - c &= 0 \end{aligned}$

c is <u>free</u> $\Rightarrow$ there are non-trivial solutions

$\Rightarrow v_1 + v_2, v_2 + v_3, v_1 - v_3$ <u>not</u> lin. indep.

and <u>not</u> a basis.

4a. (10 points) Let V be a finite dimensional vector space. Let $T: V \to V$ be a linear transformation. Assume that $\operatorname{rank}(T) = \operatorname{rank}(T \circ T)$. Prove that $R(T) \cap N(T) = \{\vec{0}\}$.

(Hints: $R(T)$ denotes the range of T, $N(T)$ the null space of T. Use the dimension formula for the restriction of T to $R(T)$.)

4b. Let $\beta = \{(-1, 1), (2, -1)\}$.

(i) (5 points) Show that β is a basis for $\mathbb{R}^2$.

(ii) (10 points) Taking β to be an ordered basis, find its dual basis $\{f_1, f_2\}$ in $(\mathbb{R}^2)^*$.

4a) Write $\mathcal{U}: R(T) \longrightarrow V, \; v \mapsto T(v)$.

Dim-formula: $\underbrace{\operatorname{rank}(\mathcal{U})}_{=\operatorname{rank}(T\circ T)} + \operatorname{nullity}(\mathcal{U}) = \operatorname{rank}(T)$.

$\Rightarrow \underbrace{\dim(R(T) \cap N(T))}_{=0 \text{ by assumption.}} = \operatorname{nullity}(\mathcal{U}) = \operatorname{rank}(T) - \operatorname{rank}(T \circ T)$

$\Rightarrow R(T) \cap N(T) = \{\vec{0}\}$ QED

4b) i) $a\begin{pmatrix}-1\\1\end{pmatrix} + b\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow \begin{matrix}-a + 2b = 0\\ a - b = 0\end{matrix} \Leftrightarrow$

$\Leftrightarrow \begin{matrix}-a + 2b = 0\\ b = 0\end{matrix} \Leftrightarrow a = b = 0.$

$\Rightarrow \left\{\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}\right\}$ is lin. indep. Since $\# = 2$, it is basis.

4b) ii) Find $f_1(x,y) = ax + by$: $\quad \begin{matrix} f_1(-1,1) = 1 \\ f_1(2,-1) = 0 \end{matrix} \Leftrightarrow$

$\Leftrightarrow \begin{matrix}-a + b = 1\\ 2a - b = 0\end{matrix} \Leftrightarrow a = 1, \; b = 2$

$\rightsquigarrow f_1(x,y) = x + 2y$

Find $f_2(x,y) = ax + by$: $\quad \begin{matrix} f_2(-1,1) = 0 \\ f_2(2,-1) = 1 \end{matrix} \Leftrightarrow$

$\Leftrightarrow \begin{matrix}-a + b = 0\\ 2a - b = 1\end{matrix} \Leftrightarrow a = b = 1$

$\rightsquigarrow f_2(x,y) = x + y.$