UH - Math 4377/6308 - Dr. Heier - Fall 2011
Sample Final Exam
Time: 175 min

WITH SOLUTION

1. (a) (3 points) Let $z = a + ib$ be a complex number. Prove that $|z|^2 = z\bar{z}$.

(b) (4 points) Solve the equation $z(1 + i) = i$ for z.

(c) (3 points) Is the function $f : (1, 4) \to (1, 2),\ x \mapsto \sqrt{x}$ one-to-one? Onto?

2. (a) (5 points) Determine if the following subset of $\mathbb{R}^2$ is a subspace. Justify your answer carefully:

$$\{(a_1, a_2) \in \mathbb{R}^2 : a_1 \cdot a_2 = 0\}.$$

(b) (5 points) Determine if the following subsets of the vector space of 2×2 matrices with real entries are subspaces. You may assume as true that the set of 2×2 matrices with real entries forms a vector space with the usual addition and scalar multiplication.

(a) $\left\{ \begin{pmatrix} a_1 & a_1 + a_2 \\ a_2 & 0 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\}$

(b) $\left\{ \begin{pmatrix} a_1 & a_1 \cdot a_2 \\ a_2 & a_3 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}$

3. (a) (5 points) Find bases for the kernel and range of

$$T : \mathbb{R}^5 \to \mathbb{R}^4, (a_1, a_2, a_3, a_4, a_5) \mapsto (a_1 + a_4 + a_5, -a_1 + a_2 + a_4, a_5 - a_4, a_1 + 2a_5).$$

(b) (5 points) Let $G = \{(1, -1, 0, 1), (1, 0, 1, 0), (1, 2, 4, 2), (0, 2, 2, 2)\}$. Let $L = \{(2, -4, -3, 0)\}$. Find a subset $H \subset G$ of cardinality 3 such that $H \cup L$ spans $\mathbb{R}^4$. Prove the spanning property with an explicit computation.

4. (a) (5 points) Find the rank of

$$\begin{pmatrix} 2 & 2 & 0 & 1 \\ 3 & 1 & 3 & 3 \\ 5 & 3 & 3 & 4 \\ 7 & 5 & 3 & 5 \\ 8 & 4 & 6 & 7 \end{pmatrix}.$$

(b) (5 points) Give an example of $A, B \in M_{4\times 4}(\mathbb{R})$ such that both A and B have rank 2, but their product AB has rank 1.

5. (10 points) Find the inverse of

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}.$$

6. (a) (5 points) Let

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 3 & 1 & 3 \\ 5 & 3 & -2 \end{pmatrix}$$

and let

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 1 & 3 \\ 4 & 2 & 1 \end{pmatrix}.$$

Find $\det(A)$, $\det(B)$ and $\det(AB)$.

(b) (5 points) Compute the determinant of

$$\begin{pmatrix} 5 & -1 & 0 & 1 \\ 4 & 0 & 1 & 0 \\ 5 & 2 & 5 & 3 \\ 4 & -4 & -3 & 0 \end{pmatrix}.$$

7. (a) (5 points) Let $A, B \in M_{n\times n}(\mathbb{R})$ be such that $AB = -BA$. Prove that if n is odd, then at least one of the two matrices A, B is not invertible.

(b) (5 points) Let $A \in M_{n\times n}(\mathbb{R})$ have two distinct eigenvalues λ_1, λ_2. Give a necessary and sufficient criterion in terms of $\dim E_{\lambda_1}$ and $\dim E_{\lambda_2}$ for the diagonalizability of A.

8. (a) (5 points) Find the eigenvalues of

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

(b) (5 points) Find the eigenvectors of A.

(c) (5 points) Find a matrix Q such that $Q^{-1}AQ$ is diagonal.

9. (15 points) Is the matrix

$$A = \begin{pmatrix} 1 & 0 & -8 \\ -4 & 9 & -4 \\ -10 & 0 & -1 \end{pmatrix}$$

diagonalizable? If yes, give a basis of eigenvectors of A for $\mathbb{R}^3$.

Solution

1. a) Left hand side: $|z|^2 = \sqrt{a^2+b^2}^{\,2} = \underline{a^2+b^2}$

Right " " : $z\bar{z} = (a+ib)(a-ib) =$

$= a^2 - iab + iab + i(-i)b^2$

$\underset{\substack{\uparrow \\ i^2=-1}}{=} \underline{a^2+b^2}$

b) $z = \dfrac{i}{1+i} = \dfrac{i(1-i)}{(1+i)(1-i)} = \dfrac{1+i}{2} = \dfrac{1}{2} + \dfrac{1}{2}i$

c) One-to-one: yes: let $\sqrt{x} = \sqrt{y}$

$\Rightarrow \sqrt{x}^2 = \sqrt{y}^2 \Rightarrow x = y$

Onto: yes: let $y \in (1,2)$. Then

$y^2 \in (1,4)$ and $\sqrt{y^2} = y$.

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2. a) $\binom{1}{0}$ and $\binom{0}{1}$ both satisfy $a_1 \cdot a_2 = 0$

but $\binom{1}{0} + \binom{0}{1} = \binom{1}{1}$ does not. $\Rightarrow$ not a subspace

b) a) Closedness for "+":

$$\begin{pmatrix} a_1 & a_1+a_2 \\ a_2 & 0 \end{pmatrix} + \begin{pmatrix} b_1 & b_1+b_2 \\ b_2 & 0 \end{pmatrix} = \begin{pmatrix} a_1+b_1 & a_1+a_2+b_1+b_2 \\ a_2+b_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_1+b_1 & (a_1+b_1)+(a_2+b_2) \\ a_2+b_2 & 0 \end{pmatrix} \checkmark$$

Closedness for "$\cdot$":

$$c\begin{pmatrix} a_1 & a_1+a_2 \\ a_2 & 0 \end{pmatrix} = \begin{pmatrix} ca_1 & c(a_1+a_2) \\ ca_2 & 0 \end{pmatrix} = \begin{pmatrix} ca_1 & ca_1+ca_2 \\ ca_2 & 0 \end{pmatrix} \checkmark$$

b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ both satisfy the condition,

but $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ does

not $\Rightarrow$ <u>not</u> closed $\Rightarrow$ <u>not</u> a subspace

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3 a) Computation yields

$$N(T) = \left\{ \begin{pmatrix} -2a_5 \\ -3a_5 \\ a_3 \\ a_5 \\ a_5 \end{pmatrix} \middle| \, a_3, a_5 \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} -2 \\ -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\underbrace{\dim(\mathbb{R}^5)}_{=5} = \underbrace{\dim N(T)}_{=2} + \dim R(T)$$

$\Rightarrow$ need 3 lin. indep. vectors to have basis for the range $R(T)$.

$$T\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\-1\\0\end{pmatrix}$$

$$a_1\begin{pmatrix}1\\-1\\0\\1\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + a_3\begin{pmatrix}1\\1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow \dots \Leftrightarrow a_1 = a_2 = a_3 = 0$$

$$\Rightarrow \left\{ \begin{pmatrix}1\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\0\end{pmatrix} \right\}$$ is basis of $R(T)$.

3b) Method: Trial and error:

I pick $H = \left\{ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\4\\2\end{pmatrix}, \begin{pmatrix}0\\2\\2\\2\end{pmatrix} \right\}$.

Check: $a_1\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + a_2\begin{pmatrix}1\\2\\4\\2\end{pmatrix} + a_3\begin{pmatrix}0\\2\\2\\2\end{pmatrix} + a_4\begin{pmatrix}2\\-4\\-3\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$$
\begin{aligned}
&a_1 + a_2 + 2a_4 = 0 \\
\Leftrightarrow\quad & 2a_2 + 2a_3 - 4a_4 = 0 \\
&a_1 + 4a_2 + 2a_3 - 3a_4 = 0 \\
& 2a_2 + 2a_3 = 0
\end{aligned}
$$

(1)

$$
\begin{aligned}
\Leftrightarrow\quad &a_2 + a_3 - 2a_4 = 0 \\
&3a_2 + 2a_3 - 5a_4 = 0
\end{aligned}
$$

(4)

(1)

(2)

$$
\begin{aligned}
\Leftrightarrow\quad &-a_3 + a_4 = 0 \\
&4a_4 = 0
\end{aligned}
$$

$$\Leftrightarrow \quad a_1 = a_2 = a_3 = a_4 = 0 \quad \checkmark$$

4a) Rank is unchanged by taking transpose:

$$
\begin{pmatrix}
2 & 3 & 5 & 7 & 8 \\
2 & 1 & 3 & 5 & 4 \\
0 & 3 & 3 & 3 & 6 \\
1 & 3 & 4 & 5 & 7
\end{pmatrix}
$$

$$
\rightsquigarrow
\begin{pmatrix}
1 & 3 & 4 & 5 & 7 \\
0 & 3 & 3 & 3 & 6 \\
0 & -3 & -3 & -3 & -6 \\
0 & -5 & -5 & -5 & -10
\end{pmatrix}
\rightsquigarrow
\begin{pmatrix}
1 & 3 & 4 & 5 & 7 \\
0 & 3 & 3 & 3 & 6 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

non-zero rows $= \underline{2 = \text{rank}}$

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4b)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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5)

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{array}\right)$$

$$\Big\downarrow$$

(it's hard not to make a computational error here, but try!)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right)$$

the inverse matrix

6 a) $\det A = 2 \cdot \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 3 \\ 5 & -2 \end{vmatrix}$

$= 2(-11) - 2(-21) = \underline{20}$

$\det B = \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 6 \\ 0 & -2 & 5 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 6 \\ 0 & 0 & -1 \end{pmatrix}$

$= \underline{2}$

$\det(AB) = \det A \cdot \det B = \underline{40}$

b) Suggested method: expansion along <u>2nd row</u>:

$\det \begin{pmatrix} 5 & -1 & 0 & 1 \\ 4 & 0 & 1 & 0 \\ 5 & 2 & 5 & 3 \\ 4 & -4 & -3 & 0 \end{pmatrix} = -4 \cdot \det \begin{pmatrix} -1 & 0 & 1 \\ 2 & 5 & 3 \\ -4 & -3 & 0 \end{pmatrix}$

$- \det \begin{pmatrix} 5 & -1 & 1 \\ 5 & 2 & 3 \\ 4 & -4 & 0 \end{pmatrix}$

$= \ldots = \underline{-40}$

7 a) $AB = -BA$

$\Rightarrow \det(AB) = \det(-BA)$

$\qquad \det A \cdot \det B \qquad \det(-B) \cdot \det A$

If $\det A = 0 \Rightarrow A$ not invertible – done.

So assume $\det A \neq 0$.

$\Rightarrow \det B = \det(-B) = (-1)^n \det B \overset{n \text{ odd}}{=} -\det B$

$\Rightarrow 2 \cdot \det B = 0 \Rightarrow \det B = 0$

$\Rightarrow B$ not invertible QED

7 b) $\dim E_{\lambda_1} + \dim E_{\lambda_2} = n.$

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8 a) $\det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 8 \overset{!}{=} 0$

$\Leftrightarrow \lambda = 2$ or $\lambda = 4$

b) Compute:

$E_2 = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

$E_4 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

c) $Q = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

9) $\det\begin{pmatrix} 1-\lambda & 0 & -8 \\ -4 & 9-\lambda & -4 \\ -10 & 0 & -1-\lambda \end{pmatrix} \overset{\text{exp. along first row}}{=}$

$= (1-\lambda)\begin{vmatrix} 9-\lambda & -4 \\ 0 & -1-\lambda \end{vmatrix} - 8\begin{vmatrix} -4 & 9-\lambda \\ -10 & 0 \end{vmatrix}$

$= (9-\lambda)\left(-(1-\lambda)(\lambda+1) - 80\right) = (9-\lambda)(\lambda^2-81)$

$= (9-\lambda)(\lambda+9)(\lambda-9) = -(\lambda-9)^2(\lambda-(-9))$

The eigenvalues are: $\lambda = 9$ (twice)

$\lambda = -9$

$E_9 = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$ (computation omitted)

$E_{-9} = \operatorname{span}\left\{\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}\right\}$ (computation omitted)

$\Rightarrow \left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}\right\}$ is a basis

of eigenvectors (and A is diagonalizable)

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