WITH SOLUTION

1. (a) (3 points) Let $z = a + ib$ be a complex number. Prove that $|z|^2 = z\bar{z}$.

(b) (4 points) Solve the equation $z(1 + i) = i$ for z.

(c) (3 points) Is the function $f : (1, 4) \to (1, 2)$, $x \mapsto \sqrt{x}$ one-to-one? Onto?

2. (a) (5 points) Determine if the following subset of $\mathbb{R}^2$ is a subspace. Justify your answer carefully:
$$\{(a_1, a_2) \in \mathbb{R}^2 : a_1 \cdot a_2 = 0\}.$$
(b) (5 points) Determine if the following subsets of the vector space of 2×2 matrices with real entries are subspaces. You may assume as true that the set of 2×2 matrices with real entries forms a vector space with the usual addition and scalar multiplication.

(a) $\left\{ \begin{pmatrix} a_1 & a_1 + a_2 \\ a_2 & 0 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\}$

(b) $\left\{ \begin{pmatrix} a_1 & a_1 \cdot a_2 \\ a_2 & a_3 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}$

3. (a) (5 points) Find bases for the kernel and range of
$$T : \mathbb{R}^5 \to \mathbb{R}^4, (a_1, a_2, a_3, a_4, a_5) \mapsto (a_1 + a_4 + a_5, -a_1 + a_2 + a_4, a_5 - a_4, a_1 + 2a_5).$$

(b) (5 points) Let $G = \{(1, -1, 0, 1), (1, 0, 1, 0), (1, 2, 4, 2), (0, 2, 2, 2)\}$. Let $L = \{(2, -4, -3, 0)\}$. Find a subset $H \subset G$ of cardinality 3 such that $H \cup L$ spans $\mathbb{R}^4$. Prove the spanning property with an explicit computation.

4. (a) (5 points) Find the rank of
$$\begin{pmatrix} 2 & 2 & 0 & 1 \\ 3 & 1 & 3 & 3 \\ 5 & 3 & 3 & 4 \\ 7 & 5 & 3 & 5 \\ 8 & 4 & 6 & 7 \end{pmatrix}.$$

(b) (5 points) Give an example of $A, B \in M_{4 \times 4}(\mathbb{R})$ such that both A and B have rank 2, but their product AB has rank 1.

5. (10 points) Find the inverse of

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}.$$

6. (a) (5 points) Let

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 3 & 1 & 3 \\ 5 & 3 & -2 \end{pmatrix}$$

and let

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 1 & 3 \\ 4 & 2 & 1 \end{pmatrix}.$$

Find $\det(A)$, $\det(B)$ and $\det(AB)$.

(b) (5 points) Compute the determinant of

$$\begin{pmatrix} 5 & -1 & 0 & 1 \\ 4 & 0 & 1 & 0 \\ 5 & 2 & 5 & 3 \\ 4 & -4 & -3 & 0 \end{pmatrix}.$$

7. (a) (5 points) Let $A, B \in M_{n \times n}(\mathbb{R})$ be such that $AB = -BA$. Prove that if n is odd, then at least one of the two matrices A, B is not invertible.

(b) (5 points) Let $A \in M_{n \times n}(\mathbb{R})$ have two distinct eigenvalues λ_1, λ_2. Give a necessary and sufficient criterion in terms of $\dim E_{\lambda_1}$ and $\dim E_{\lambda_2}$ for the diagonalizability of A.

(c) (5 points) Let V be a vector space over $\mathbb{R}$. Let $Z : V \to V, v \mapsto \vec{0}$ be the zero transformation. Let $P : V \to V$ be such that $P \circ P = Z$. Let $I : V \to V$ be the identity map. Prove that $cI - P$ is invertible for all $c \in \mathbb{R} \setminus \{0\}$.

8. (a) (3 points) Find the eigenvalues of

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

(b) (5 points) Find the eigenvectors of A.

(c) (2 points) Find a matrix Q such that $Q^{-1}AQ$ is diagonal.

9. (15 points) Is the matrix
$$A = \begin{pmatrix} 1 & 0 & -8 \\ -4 & 9 & -4 \\ -10 & 0 & -1 \end{pmatrix}$$
diagonalizable? If yes, give a basis of eigenvectors of A for $\mathbb{R}^3$.

Solution

1. a) Left hand side: $|z|^2 = \sqrt{a^2+b^2}^2 = \underline{a^2+b^2}$

   Right " " : $z\bar{z} = (a+ib)(a-ib) =$

   $$= a^2 - iab + iab + i(-i)b^2$$

   $$\underset{\underset{i^2=-1}{\uparrow}}{=} \underline{a^2+b^2}$$

   b) $z = \dfrac{i}{1+i} = \dfrac{i(1-i)}{(1+i)(1-i)} = \dfrac{1+i}{2} = \dfrac{1}{2} + \dfrac{1}{2}i$

   c) One-to-one: <u>yes</u>: let $\sqrt{x} = \sqrt{y}$

   $$\Rightarrow \sqrt{x}^2 = \sqrt{y}^2 \Rightarrow x = y$$

   Onto: <u>yes</u>: let $y \in (1,2)$. Then

   $$y^2 \in (1,4) \text{ and } \sqrt{y^2} = y.$$

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2. a) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ both satisfy $a_1 \cdot a_2 = 0$

   But $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ does not. $\Rightarrow$ <u>not</u> a subspace

   b) a) Closedness for "+":

   $$\begin{pmatrix} a_1 & a_1+a_2 \\ a_2 & 0 \end{pmatrix} + \begin{pmatrix} b_1 & b_1+b_2 \\ b_2 & 0 \end{pmatrix} = \begin{pmatrix} a_1+b_1 & a_1+a_2+b_1+b_2 \\ a_2+b_2 & 0 \end{pmatrix}$$

   $$= \begin{pmatrix} a_1+b_1 & (a_1+b_1)+(a_2+b_2) \\ a_2+b_2 & 0 \end{pmatrix} \checkmark$$

   $\boxed{1}$

Closedness for ".":

$$c \begin{pmatrix} a_1 & a_1+a_2 \\ a_2 & 0 \end{pmatrix} = \begin{pmatrix} c a_1 & c(a_1+a_2) \\ c a_2 & 0 \end{pmatrix} = \begin{pmatrix} c a_1 & c a_1 + c a_2 \\ c a_2 & 0 \end{pmatrix} \checkmark$$

b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ both satisfy the condition,

but $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ does

not $\Rightarrow$ <u>not</u> closed $\Rightarrow$ <u>not</u> a subspace

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3 a) Computation yields

$$N(T) = \left\{ \begin{pmatrix} -2a_5 \\ -3a_5 \\ a_3 \\ a_5 \\ a_5 \end{pmatrix} \;\middle|\; a_3, a_5 \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{pmatrix} -2 \\ -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\dim(\mathbb{R}^5) = \dim N(T) + \dim R(T)$$
$$\underset{5}{\overset{\|}{}} \qquad\qquad \underset{2}{\overset{\|}{}}$$

$\Rightarrow$ need 3 lin. indep. vectors to have basis for the range $R(T)$.

$\sqrt{2}$

$$T\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\-1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\0\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\-1\\-1\\0\end{pmatrix}$$

$$a_1\begin{pmatrix}1\\-1\\0\\1\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + a_3\begin{pmatrix}1\\-1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$(\Leftarrow) \ \dots \ (\Leftarrow) \quad a_1 = a_2 = a_3 = 0$$

$$\Rightarrow \left\{ \begin{pmatrix}1\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\\0\end{pmatrix} \right\} \text{ is basis of } R(T)$$

3b) Method: Trial and error:

I pick $H = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\4\\2\end{pmatrix}, \begin{pmatrix}0\\2\\2\\2\end{pmatrix} \right\}$.

Check: $a_1\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + a_2\begin{pmatrix}1\\2\\4\\2\end{pmatrix} + a_3\begin{pmatrix}0\\2\\2\\2\end{pmatrix} + a_4\begin{pmatrix}2\\-4\\-3\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$$\boxed{3}$$

$$(=) \quad \begin{aligned} a_1 + a_2 \quad\quad + 2a_4 &= 0 \\ 2a_2 + 2a_3 - 4a_4 &= 0 \\ a_1 + 4a_2 + 2a_3 - 3a_4 &= 0 \\ 2a_2 + 2a_3 \quad\quad &= 0 \end{aligned}$$

$$(=) \quad \begin{aligned} & ① \\ & a_2 + a_3 - 2a_4 = 0 \\ & 3a_2 + 2a_3 - 5a_4 = 0 \\ & ④ \end{aligned}$$

$$(=) \quad \begin{aligned} & ① \\ & ② \\ & -a_3 + a_4 = 0 \\ & 4a_4 = 0 \end{aligned}$$

$$(=) \quad a_1 = a_2 = a_3 = a_4 = 0 \quad \checkmark$$

4a) Rank is unchanged by taking transpose:

$$\begin{pmatrix} 2 & 3 & 5 & 7 & 8 \\ 2 & 1 & 3 & 5 & 4 \\ 0 & 3 & 3 & 3 & 6 \\ 1 & 3 & 4 & 5 & 7 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 3 & 4 & 5 & 7 \\ 0 & 3 & 3 & 3 & 6 \\ 0 & -3 & -3 & -3 & -6 \\ 0 & -5 & -5 & -5 & -10 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 4 & 5 & 7 \\ 0 & 3 & 3 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

\# non-zero rows = $\underline{2 = \text{rank}}$

$\boxed{4}$

4b)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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5)

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{array}\right)$$

$\Big\{ \Big\downarrow$ (it's hard not to make a computational error here, but try!)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right)$$

$\underbrace{\hphantom{xxxxxxxxxxxxxx}}$

the inverse matrix

6a) $\det A = 2 \cdot \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 3 \\ 5 & -2 \end{vmatrix}$

$= 2(-11) - 2(-21) = \underline{20}$

$\det B = \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 6 \\ 0 & -2 & 5 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 6 \\ 0 & 0 & -1 \end{pmatrix}$

$= \underline{2}$

$\det(AB) = \det A \cdot \det B = \underline{40}$.

b) Suggested method: expansion along **2nd row**:

$\det \begin{pmatrix} 5 & -1 & 0 & 1 \\ 4 & 0 & 1 & 0 \\ 5 & 2 & 5 & 3 \\ 4 & -4 & -3 & 0 \end{pmatrix} = -4 \cdot \det \begin{pmatrix} -1 & 0 & 1 \\ 2 & 5 & 3 \\ -4 & -3 & 0 \end{pmatrix}$

$- \det \begin{pmatrix} 5 & -1 & 1 \\ 5 & 2 & 3 \\ 4 & -4 & 0 \end{pmatrix}$

$= \ldots = \underline{-40}$

$\boxed{6}$

7a) $AB = -BA$

$\Rightarrow \quad \det(AB) = \det(-BA)$

$\underbrace{\det A \cdot \det B}_{} \qquad \underbrace{\det(-B) \cdot \det A}_{}$

If $\det A = 0 \Rightarrow A$ not invertible — done.

So assume $\det A \neq 0$.

$\Rightarrow \quad \det B = \det(-B) = (-1)^n \det B \overset{n \text{ odd}}{=} -\det B$

$\Rightarrow \quad 2 \cdot \det B = 0 \Rightarrow \det B = 0$

$\Rightarrow \quad B$ not invertible QED

7b) $\dim E_{\lambda_1} + \dim E_{\lambda_2} = n$

8a) $\det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 8 \overset{!}{=} 0$

$\Rightarrow \lambda = 2$ or $\lambda = 4$

b) Compute:

$E_2 = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

$E_4 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

c) $Q = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

7c) Observe that
$(cI - P) \circ (cI + P) =$
$c^2 I - P \circ P = c^2 I - Z = \underline{c^2 I}$
$\Rightarrow cI - P$ onto

By the dimension formula, $cI - P$ is also one-to-one.
$\Rightarrow cI - P$ is invertible, QED

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g)

$$\det \begin{pmatrix} 1-\lambda & 0 & -8 \\ -4 & 9-\lambda & -4 \\ -10 & 0 & -1-\lambda \end{pmatrix} \overset{\text{exp. along first row}}{=}$$

$$= (1-\lambda) \begin{vmatrix} 9-\lambda & -4 \\ 0 & -1-\lambda \end{vmatrix} - 8 \begin{vmatrix} -4 & 9-\lambda \\ -10 & 0 \end{vmatrix}$$

$$= (9-\lambda)\left(-(1-\lambda)(\lambda+1) - 80\right) = (9-\lambda)(\lambda^2-81)$$

$$= (9-\lambda)(\lambda+9)(\lambda-9) = -(\lambda-9)^2(\lambda-(-9))$$

The eigenvalues are: $\lambda = 9$ (twice)

$$\lambda = -9$$

$$E_9 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{(computation omitted)}$$

$$E_{-9} = \text{span}\left\{ \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} \right\} \quad \text{(computation omitted)}$$

$$\Rightarrow \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} \right\} \text{ is a basis}$$

of eigenvectors (and A is diagonalizable)