UH - Math 6302 (Modern Algebra) - Dr. Heier - Fall 2013

Midterm Exam

Wednesday, October 23, 2013

Print your NAME: Solution

Solve all of the 6 problems. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit. This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. The time allowed will be announced by the proctor.

Problem 1 ______	/10 points
Problem 2 ______	/30 points
Problem 3 ______	/15 points
Problem 4 ______	/15 points
Problem 5 ______	/15 points
Problem 6 ______	/15 points
Total ______	/100 points

1. (10 points) Let H and K be subgroups of the group G. Prove that $H \cup K$ is a subgroup of G if and only if $H \subset K$ or $K \subset H$.

"$\Leftarrow$" W.l.o.g, assume $H \subset K$.

$\Rightarrow$ $H \cup K = K$ which is a subgroup by assumption.

"$\Rightarrow$" Assume $H \not\subset K$ and $K \not\subset H$ (and derive contradiction)

Let $h \in H \setminus K$ and $k \in K \setminus H$ (exist!)

Since $h, k \in H \cup K =$ subgp, we have $h \cdot k \in H \cup K$.

Case 1 $h \cdot k \in H \Rightarrow k \in h^{-1} H = H$ ↯

Case 2 $h \cdot k \in K \Rightarrow h \in K k^{-1} = K$ ↯.

2. (10 points PER ITEM) Determine if the following statements are TRUE or FALSE. In each case, provide a PROOF for your answer.

(a) There exists a group of order 10 and a set A of cardinality 8 such that G acts transitively on A.

(b) There exists a group of order 10 and a set A of cardinality 5 such that G acts transitively on A.

(c) There exists an abelian group of order 12 which acts faithfully and transitively on a set A of cardinality 6.

a) False. Proof: let $a \in A$. Transitivity $\Rightarrow Ga = A$.

From class: $8 = \#A = \#Ga = |G : G_a| = \frac{\#G}{\#G_a} = \frac{10}{\#G_a}$.

$\#G_a$ is an integer. ↯.

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b) True. let $G = \mathbb{Z}_2 \times \mathbb{Z}_5$, $A = \mathbb{Z}_5$.

The map $\begin{aligned} G \times A &\to A \\ ((x,y), z) &\mapsto y+z \end{aligned}$ is the required group action.

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c) Let $a \in A$. Transitivity $\Rightarrow 6 = \frac{12}{\#G_a}$

$\Rightarrow \#G_a = 2$ and $G_a = \{e, x\}$.

let $b \in A$ arbitrary. Transitivity $\Rightarrow \exists g \in G : b = ga$.

Note: $xb = x(ga) = (xg)a \overset{G \text{ abelian}}{=} (gx)a = g(xa) = ga = b$

$\Rightarrow x \in G_b$

Since b was arbitrary, the action is not faithful. ↯.

3. (a) (5 points) Let H and K be finite subgroups of the group G. Assume that the orders of H and K are relatively prime. Prove that $H \cap K = \{e\}$.

(b) (10 points) Let G be a group. Let H be a subgroup of order 2 in G. Prove that $N_G(H) = C_G(H)$.

a) Let $x \in H \cap K$. Lagrange $\Rightarrow$ $\operatorname{ord}(x) \mid \#H$ and $\operatorname{ord}(x) \mid \#K$.

$\gcd(\#H, \#K) = 1$ by assumption $\Rightarrow$ $\operatorname{ord}(x) = 1$

$\Rightarrow x = e$.

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b) "$\supseteq$" trivial

"$\subseteq$" Let $g \in N_G(H)$. Write $H = \{e, x\}$.

Clear: $geg^{-1} = e$.

Still have to prove: $gxg^{-1} = x$.

If not, then $g \in N_G(H) \Rightarrow gxg^{-1} = e$

$\Rightarrow gx = g \Rightarrow x = e$ ↯.

4. (15 points) Determine explicitly the set $Syl_3(S_4)$. Hint: Use Sylow's theorem and additional explicit considerations.

Let $n_3 = \# Syl_3(S_4)$. Note: $\# S_4 = 4! = 2^3 \cdot 3$.

Sylow's Thm $\Rightarrow$ $n_3 \equiv 1 \mod 3$ and $n_3 | 8$.

$\Rightarrow$ $n_3 = 1$ or $n_3 = 4$.

Observe: $\langle (1\ 2\ 3) \rangle = \{e, (1\ 2\ 3), (3\ 2\ 1)\}$
$\cong \mathbb{Z}_3$
is a 3-Sylow subgroup of S_4.

So are $\langle (1\ 2\ 4) \rangle$, $\langle (2\ 3\ 4) \rangle$, $\langle (1\ 3\ 4) \rangle$.

$\Rightarrow$ $n_3 = 4$ and $Syl_3(S_4) =$

$\{ \langle (1\ 2\ 3) \rangle, \langle (1\ 2\ 4) \rangle, \langle (1\ 3\ 4) \rangle, \langle (2\ 3\ 4) \rangle \}$

5. (a) (5 points) Are the groups $\mathbb{Z}_{60} \times \mathbb{Z}_{45} \times \mathbb{Z}_{12} \times \mathbb{Z}_{36}$ and $\mathbb{Z}_{15} \times \mathbb{Z}_{20} \times \mathbb{Z}_{27} \times \mathbb{Z}_9 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ isomorphic? Justify your answer carefully.
(b) (10 points) Give a complete list of all abelian groups of order $108 = 2^2 \cdot 3^3$.

a) $$\underbrace{\mathbb{Z}_{60}}_{\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5} \times \underbrace{\mathbb{Z}_{45}}_{\mathbb{Z}_9 \times \mathbb{Z}_5} \times \underbrace{\mathbb{Z}_{12}}_{\mathbb{Z}_3 \times \mathbb{Z}_4} \times \underbrace{\mathbb{Z}_{36}}_{\mathbb{Z}_4 \times \mathbb{Z}_9} \cong \quad (*)$$

$$\underbrace{\mathbb{Z}_{15}}_{\mathbb{Z}_3 \times \mathbb{Z}_5} \times \underbrace{\mathbb{Z}_{20}}_{\mathbb{Z}_4 \times \mathbb{Z}_5} \times \mathbb{Z}_{27} \times \mathbb{Z}_9 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \cong \quad (**)$$

$$\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_{27} \times \mathbb{Z}_9 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \quad (**)$$

The second version of the Fundamental Thm for abelian groups says: Since the $\mathbb{Z}_{27}$ from $(**)$ is not present in $(*)$, the groups are <u>NOT</u> isomorphic.

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b) First version of the Fund. Thm:

$\mathbb{Z}_{108}$, $\mathbb{Z}_{54} \times \mathbb{Z}_2$, $\mathbb{Z}_{36} \times \mathbb{Z}_3$, $\mathbb{Z}_{18} \times \mathbb{Z}_6$,
$\mathbb{Z}_{12} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_3$.

(Solution via the Second Version:
$\mathbb{Z}_4 \times \mathbb{Z}_{27}$, $\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{27}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$)

6. (15 points) Let H and K be groups. Let $\varphi : K \to Aut(H)$ be a homomorphism. Let $\sigma : K \to K$ be an automorphism of K. Let $\psi = \varphi \circ \sigma$. Prove that the semi-direct products $H \rtimes_\varphi K$ and $H \rtimes_\psi K$ are isomorphic.

Let $f: H \rtimes_\psi K \longrightarrow H \rtimes_\varphi K$

$$(h, k) \longmapsto (h, \sigma(k)).$$

Since σ is bijective, f is clearly bijective also.
It remains to check the homomorphism property.

$$f((h_1, k_1)(h_2, k_2)) =$$

$$= f((h_1(\psi(k_1))(h_2), k_1 k_2) = (h_1(\psi(k_1))(h_2), \sigma(k_1 k_2)) =$$

$$= (h_1((\varphi(\sigma(k_1))))(h_2), \sigma(k_1)\sigma(k_2))$$

$\psi = (\varphi \circ \sigma)$;
σ is automorphism

$$= (h_1, \sigma(k_1))(h_2, \sigma(k_2))$$

$$= f(h_1, k_1) \cdot f(h_2, k_2) \quad \text{QED}$$