Selected Solutions for HW11

#3 §4.3 Problem 17: Let $A, B \in M_{n \times n}(F)$ with $AB = -BA$. Prove that if n is odd and F is not a field of characteristic 2 then either A or B is not invertible.

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proof: I will show a contradiction.

Assume that A and B are both invertible. Then $|A| \neq 0$ and $|B| \neq 0$.

We have that $\det(AB) = \det(A)\det(B)$

$$\det(-BA) = (-1)^n \det(BA) = (-1)^n \det(B)\det(A) = (-1)^n \det(A)\det(B).$$

Since $AB = -BA$ we have the following:

$$\det(A)\det(B) = (-1)^n \det(A)\det(B)$$; since $\det(A) \neq 0$, $\det(B) \neq 0$ we can divide.

$\underline{1 = (-1)^n}$ but this is a contradiction because n is odd and $1 \neq -1$. $\therefore$ either one (or both) of A and B must be non-invertible. Q.E.D.

#6 (b): Want to find the eigenvalues, eigenvectors, basis, Q, D, of Matrix A.

§5.1 Problem 3 (b)

$$A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} ; \quad (A - \lambda I) = \begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix}$$

; want to find the determinant of this matrix. I'll expand down the first column.

$$-\lambda\left[(1-\lambda)(5-\lambda) + 2\right] + \left[-2(5-\lambda) + 6\right] + 2\left[2 + 3(1-\lambda)\right]$$

* move and combine and all that algebraic stuff that I can skip.

$$\det(A - \lambda I) = (3-\lambda)(2-\lambda)(1-\lambda) = 0 \Rightarrow \boxed{\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3}$$

$\rightarrow$ Cont.

Now we find the ~~eigen~~ vectors that define the Null space of $(A - \lambda_i I)$

so for $\lambda = 1$

$$(A - I) = \begin{pmatrix} -1 & 2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From here you can use gaussian elimination or ansatz type approach

For $\lambda = 1$: $\vec{x}_1 = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ $\forall\ t \in \mathbb{R}$.

For $\lambda = 3$: $\vec{x}_3 = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\forall\ t \in \mathbb{R}$.

For $\lambda = 2$: $\vec{x}_2 = t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\forall\ t \in \mathbb{R}$

$\{x_1, x_2, x_3\}$ is the set of eigenvectors.

$$\text{Basis} = \text{span}\left(\{x_1, x_2, x_3\}\right).$$

$$Q = \begin{array}{c} \begin{matrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{matrix} \\ \begin{pmatrix} -1 & 1 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

We see that Q is invertible since it has full rank.

#8 § 5.2 Problem 2(d): This is very similar to the above but we have an algebraic multiplicity of two so we need to make sure that the dimensionality is correct for A to be deemed diagonalizable.

$$A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix} \text{ so } (A - \lambda I) = \begin{pmatrix} 7-\lambda & -4 & 0 \\ 8 & -5-\lambda & 0 \\ 6 & -6 & 3-\lambda \end{pmatrix}$$

; Find this determinant and set $= 0$ to solve characteristic polynomial to find eigenvalues.

$\longrightarrow$

#6 cont... I will skip these steps and present the correct final form of the factored characteristic polynomial. pg 3

$$\det(A-\lambda I) = (\lambda-3)(\lambda-3)(\lambda+1) = 0 \Rightarrow \boxed{\lambda_1 = -1,\ \lambda_2 = 3,\ \lambda_3 = 3.}$$

Now we see the multiplicity is two for $\lambda = 3$.

we need $2 = n - \text{rank}(A - \lambda I)$

$$2 = 3 - \text{rank}(A - 3I)$$

$$A - 3I = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \end{array}$$; we see that $v_1 = -v_2$ and $v_3 \equiv 0$

So the rank of this matrix is one

So $2 = 3 - 1 = 2$

$2 \overset{\checkmark}{=} 2$, good, then this matrix is diagonalizable. ($\lambda=-1$ will give no issues)

The eigenvectors for $\lambda = 3$ are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The eigenvector for $\lambda = -1$ is $\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$

So $Q = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ with $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

#2 § 4.3 Problem 7: Cramer's Rule and determinant Calculation.

I just want to note that too many students make little mistakes when calc. determinants. Make sure the minus signs are accounted for and go along the row or column that will make it easiest on you! Just remember that $(-1)^{i+j}$ in the expansion,

$3\times3 = \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ $4\times4 = \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$ and so on.

OR minor Gaussian elimination at first could help with work on the back end.