Math 4377/6308 (Dr. G. Heier)
Fall 2017, Univ. of Houston
HW solutions: HW 1

HW 1, Problem 1

$A \cup B = \{1, 2, 4, 5\}$

$A \cap B = \{5\}$

$$\begin{aligned} A \cap (B \cup C) &= \{1, 2, 5\} \cap (\{4, 5\} \cup \{4, 6\}) \\ &= \{1, 2, 5\} \cap \{4, 5, 6\} \\ &= \{5\} \end{aligned}$$

$A \cap B \cap C = \emptyset$

$$A \times C = \{(1,4), (1,6), (2,4), (2,6), (5,4), (5,6)\}$$

$\square$

HW 1, Problem 2

a) Not an equiv. rel.

Reason: not symmetric: $1 \sim 20$ is true b/c
$1 - 20 = -19 < 10$

but $20 \sim 1$ is false b/c
$20 - 1 = 19 \nless 10$.

b) Not an equiv. rel.

not transitive: $1 \sim 0$ b/c $1 \cdot 0 = 0 \geq 0$
$0 \sim -1$ b/c $0 \cdot (-1) = 0 \geq 0$

but $1 \sim (-1)$ is false b/c
$1 \cdot (-1) = -1 \ngeq 0$.

c) Yes.

Reflexive: $\forall x \in \mathbb{Z}$: $x - x = 0$ even ✓

Symm.: Let $x - y = 2 \cdot \underset{\substack{\uparrow \\ \mathbb{Z}}}{k}$. Then $y - x = 2 \cdot (\underset{\substack{\uparrow \\ \mathbb{Z}}}{-k})$ ✓

Trans.: Let $x \sim y, y \sim z \Rightarrow x - y = 2 \cdot k, \; y - z = 2 \cdot \ell$
$\Rightarrow x - z = 2 \cdot (\underset{\substack{\uparrow \\ \mathbb{Z}}}{k + \ell})$ ✓ $\square$

p. 2

HW 1, Problem 3

a) No Counterexample: $A = \{1\}$
$B = \{1, 2\}$,
$C = \{1\}$

Let $f: A \to B$, $1 \mapsto 1$
$g: B \to C$ $\left.\begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto 1 \end{array}\right\}$ Not injective

But $g \circ f: A \to C$, $1 \mapsto 1$ is injective

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b) No Same counter-example as in a).
$g \circ f$ is surjective, but f is not.

$\square$

HW 1, Problem 4

a) domain $= \{0, 1, 2, 3, 4\}$

codomain $= \mathbb{N}$

range$(f) = \{0, 2, 10, 30, 68\}$

b) Yes. Proof is by inspection:

$f(x) = f(y) \Rightarrow x = y$ is true

c) No. Clearly, $1 \notin$ range(f), but

$1 \in$ codomain

$\square$

HW 1, Problem 5

The statement claims that $a(-b)$ is the additive inverse of ab. In order to verify this, we have to check if it works as such.

$$ab + a(-b) \underset{\text{Distributive Law}}{=} a(b+(-b))$$

$$= a \cdot 0$$

$$\underset{\text{shown in class}}{=} 0 \quad \checkmark$$

$\square$

HW 1, Problem 6

$\mathbb{F}$

Since $\{x+y\sqrt{3} \mid x,y\in\mathbb{Q}\} \subset \mathbb{R}$, we have

- comm. ✓
- assoc. ✓
- distrib. ✓

Also, $0+0\cdot\sqrt{3} = 0 \in \mathbb{F}$ is neutral for $+$

$1+0\cdot\sqrt{3} = 1 \in \mathbb{F}$ is neutral for $\cdot$

Remains to show: If $x+y\sqrt{3} \in \mathbb{F}$, $\neq 0$ then

$\frac{1}{x+y\sqrt{3}} \in \mathbb{F}$.

Proof. $\frac{1}{x+y\sqrt{3}} = \frac{x-y\sqrt{3}}{(x+y\sqrt{3})(x-y\sqrt{3})} =$

$$= \frac{x-y\sqrt{3}}{x^2-3y^2} = \underbrace{\frac{x}{x^2-3y^2}}_{\in\mathbb{Q}} + \underbrace{\frac{-y}{x^2-3y^2}}_{\in\mathbb{Q}}\sqrt{3}$$

Note: $x^2-3y^2=0 \iff 3=\frac{x^2}{y^2} \iff \sqrt{3} = \pm\frac{x}{y} \in \mathbb{Q}$

Contradiction, b/c $\sqrt{3}$ not rational. $\square$

HW 1, Problem 7

$$\bar{z} = \underline{1-3i}$$

$$z+w = 1+3i+1-2i = \underline{2+i}$$

$$z \cdot w = (1+3i)(1-2i) = 1-2i+3i-6i^2$$

$$= \underline{7+i}$$

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{1+9} = \sqrt{10}$$

$$\frac{1}{z} = \frac{1}{1+3i} = \frac{1}{1+3i} \cdot \frac{1-3i}{1-3i} = \frac{1-3i}{1-(3i)^2}$$

$$= \frac{1-3i}{1+9} = \underline{\frac{1}{10} - \frac{3}{10}i}$$

HW 1, Problem 8

Solve $\quad z^2 - 2z + 5 = 0$

$\Leftrightarrow \quad z^2 - 2z + 1 + 4 = 0 \qquad$ (complete the square!)

$\Leftrightarrow \quad (z-1)^2 = -4$

$\Leftrightarrow \quad z - 1 = \pm\sqrt{-4} = \pm 2\sqrt{-1} = \pm 2i$

$\Leftrightarrow \quad \underline{z = 1 + 2i}$ or $\underline{z = 1 - 2i}$

HW 1, Problem 9 (extra credit problem)

Refl.: $x \sim x \iff x + 4x$ is multiple of 5

$\iff 5x$ is multiple of 5 ✓

Symm: Assume $x \sim y$. Then

$y + 4x = 5 \cdot k$ for some $k \in \mathbb{Z}$

This is equivalent to

$y + 4x - 5(x+y) = 5 \cdot k_1$ for some $k_1 \in \mathbb{Z}$.

$\iff -4y - x = 5 \cdot k_1$ for some $k_1 \in \mathbb{Z}$

$\iff x + 4y = 5\underbrace{(-k_1)}_{\in \mathbb{Z}}$ for some $k_1 \in \mathbb{Z}$

Trans. Assume $x \sim y$ and $y \sim z$.

$\Rightarrow \exists k, l \in \mathbb{Z}: \; y + 4x = 5 \cdot k, \; z + 4y = 5 \cdot l$

$\Rightarrow \quad " \quad : z + 4x + 5y = 5(k+l)$

$\Rightarrow \quad " \quad : z + 4x = 5\underbrace{(k+l-y)}_{\in \mathbb{Z}} \Rightarrow x \sim z$

$\square$