HW 3: Solutions to Selected Problems:

#2: To show that a subset of a V.S. is a subspace we just need to show that that subset is closed under addition and scalar multiplication and non-empty.

(a) $W_1 = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & 0 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\} \neq \emptyset$ obviously

(i) Closed under addition: given $X, Y \in W_1$. $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & 0 \end{pmatrix}$

Is $X+Y \in W_1$? $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & 0 \end{pmatrix}$

$$X+Y = \begin{pmatrix} x_1 & x_2 \\ x_3 & 0 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ y_3 & 0 \end{pmatrix}$$

$$X+Y = \begin{pmatrix} x_1+y_1 & x_2+y_2 \\ x_3+y_3 & 0 \end{pmatrix} \in W_1 \text{ since } \begin{Bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{Bmatrix} \in \mathbb{R}$$

(ii) Closed under scalar multiplication: given above and $c \in \mathbb{R}$.

Is $cX \in W_1$?

$$cX = c\begin{pmatrix} x_1 & x_2 \\ x_3 & 0 \end{pmatrix} = \begin{pmatrix} cx_1 & cx_2 \\ cx_3 & 0 \end{pmatrix} \in W_1 \text{ since } \{cx_1, cx_2, cx_3\} \in \mathbb{R}$$

(b) $W_2 = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1^2 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}$

No! Not closed under addition.

Counterexample:
$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{W_2} + \underbrace{\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}}_{W_2} = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix} \notin W_2$$

because $5 \neq 3^2 = 9$

#3: V is V.S. of all functions $f: \mathbb{R} \to \mathbb{R}$.

let $W_1 = \{ f: \mathbb{R} \to \mathbb{R} \mid f(5) = 0 \}$. W_1 is the set of functions that map each $\mathbb{R}$ to itself w/ the exception $f(5) = 0$.

Prove that W_1 is a subspace.

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as with #2, it is sufficient to show that W_1 is closed under addition and, closed under scalar mult.

(i) let $f, g \in W_1$, $c \in \mathbb{R}$

$$f(5) + g(5) = 0 + 0 \Rightarrow f + g \in W_1$$

(ii) $c f(5) \overset{\forall c}{=} c \cdot 0 = 0 \Rightarrow cf \in W_1$

$\therefore$ W_1 is a subspace of V.

Now we want to find another subset of V that is itself a subspace of V and that together with W_1, the following is true, $W_1 \oplus W_2 = V$.

let $W_2 = \{ f: \mathbb{R} \to \mathbb{R} \mid f(x) = 0 \ \forall x \in \mathbb{R} \setminus \{5\} \}$, basically allowing x to be any $\mathbb{R}$ value except for 5.

We can see that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = \{ \mathbb{R} \mapsto \mathbb{R} \}$ $\therefore$ $W_1 \oplus W_2 = V$.

#6: §1.3 #28: Prove that the set of Skew-symmetric matrices ($n \times n$ under generic field F) W_1 is a subspace of $M_{n\times n}(F)$ matrices.

[note: Skew-symmetric matrices take the form $M^t = -M$.]

Ex in $\mathbb{R}$: $\begin{pmatrix} a & b \\ b & c \end{pmatrix}^t = \begin{pmatrix} -a & -b \\ -b & -c \end{pmatrix}$

But our manipulation doesn't need to consider the actual values inside the matrices.

(i) Closed under addition:

let $X, Y \in W_1$, so $X^t = -X$; $Y^t = -Y$

$(X+Y)^t = X^t + Y^t = -X + -Y = -(X+Y)$

$\Rightarrow X+Y \in W_1$

(ii) Closed under scalar multiplication:

let $X \in W_1$, $c \in F$.

$(cX)^t = cX^t = c(-X) = -cX$

$\Rightarrow cX \in W_1$

So W_1 is a subspace of $M_{n\times n}(F)$.

Let W_2 be the subspace of $n\times n$ Symmetric matrices. $N^t = N$

Prove that $W_1 \oplus W_2 = M_{n\times n}(F)$.

First, we cleverly rewrite M as a sum of two parts and show that those parts are a symmetric part and a skew symmetric part:

$$M = \frac{1}{2}(M+M^t) + \frac{1}{2}(M-M^t).$$

Is $M+M^t \in W_2$?

$(M+M^t)^t = M^t + (M^t)^t = M + M^t$

$\Rightarrow M+M^t \in W_2$

Is $M-M^t \in W_1$?

$(M-M^t)^t = M^t - (M^t)^t = M^t - M = -(M-M^t)$

$\Rightarrow M-M^t \in W_1$

$\therefore M = W_1 + W_2$

Now that $M = W_1 + W_2$ we need to show that $W_1 \cap W_2 = \emptyset$

given $Z \in W_1 \cap W_2 \Rightarrow Z \in W_1$ and $Z \in W_2$

$Z \in W_1 \Rightarrow Z^t = -Z$
$Z \in W_2 \Rightarrow Z^t = Z$
$\Rightarrow Z = 0$

$\therefore W_1 + W_2 = M_{n\times n}(F)$ and $W_1 \cap W_2 = \{\}$

So $W_1 \oplus W_2 = M_{n\times n}(F)$. QED