Selected Solutions to HW #4

§1.4 (b) and (c). Matrices haven't yet been introduced so we'll solve these by typical algebraic manipulations.

(b) $3x_1 - 7x_2 + 4x_3 = 10$ (i)
$x_1 - 2x_2 + x_3 = 3$ (ii)
$2x_1 - x_2 - 2x_3 = 6$ (iii)

Let's create new equations by combining multiples of (i), (ii), (iii).

- $-2(ii) + (iii) = 3x_2 - 4x_3 = 0 \Rightarrow x_2 = \frac{4}{3}x_3$; plug this into other new equation.
- $-3(ii) + (i) = -x_2 + x_3 = 1$

$$-\frac{4}{3}x_3 + x_3 = 1$$
$$-\frac{1}{3}x_3 = 1$$
$$\Rightarrow x_3 = -3 \quad \text{then } x_2 = -4$$

now plug these into (ii) to find x_1

$$x_1 = 3 - 8 + 3$$
$$\Rightarrow x_1 = -2$$

$$\boxed{\text{Solution} = \begin{cases} x_1 = -2 \\ x_2 = -4 \\ x_3 = -3 \end{cases}}$$

(c) $x_1 + 2x_2 - x_3 + x_4 = 5$ (i)
$x_1 + 4x_2 - 3x_3 - 3x_4 = 6$ (ii)
$2x_1 + 3x_2 - x_3 + 4x_4 = 8$ (iii)

- $(ii) - (i) := 2x_2 - 2x_3 - 4x_4 = 1$ (iv)
- $(iii) - 2(i) := -x_2 + x_3 + 2x_4 = -2$ (v)

- $(iv) + 2(v) := 0 = -3$, but $0 \neq -3$ so this is a contradiction so this system of equations has NO solutions.

§1.4 #3(c): Determine whether the first vector in the set of vectors can be written as a combination of the other two.

$\{(3,4,1), (1,-2,1), (-2,-1,1)\}$

This question is very similar to #2. We just need to know how to put, or represent, this set of vectors as a system of equations.

$(3,4,1) = a(1,-2,1) + b(-2,-1,1)$ can be written as

(i) $a - 2b = 3$
(ii) $-2a - b = 4$
(iii) $a + b = 1$

Note: If this system has solutions then one equation needs to be a linear combination of the other two.

• $2(i) + (ii) := -5b = 10 \Rightarrow b = -2$ then by (ii) $a = -1$

So now let's make sure that these values satisfy (iii). If they do then this system is consistent and $\left\{\begin{matrix} a=-1 \\ b=-2 \end{matrix}\right\}$ is indeed the solution.

plug $a=-1$, $b=-2$ into (iii)

$-1-2=-3 \neq 1$ $\therefore$ $(3,4,1)$ cannot be expressed as a linear combination of $(1,-2,1)$ and $(-2,-1,1)$.

§ 1.6 #7: The following set of vectors span $\mathbb{R}^3$

$S = \{ \overset{u_1}{(2,-3,1)}, \overset{u_2}{(1,4,-2)}, \overset{u_3}{(-8,12,-4)}, \overset{u_4}{(1,37,-17)}, \overset{u_5}{(-3,-5,8)} \}$

Find a subset of S that is a basis for $\mathbb{R}^3$.

All we must do to find the basis is isolate the three vectors in S which are independent.

- We can immediately see that $-4u_1 = u_3$. So throw out u_3.
- We see that u_1 and u_2 are independent. Now we must check whether u_4 or u_5 is the final vector composing the basis for $\mathbb{R}^3$
- We can use the same techniques employed in the previous questions. Let's check u_4.

$$\left.\begin{aligned} &\text{(i)} & 2a + b &= 1 \\ &\text{(ii)} & -3a + 4b &= 37 \\ & & a - 2b &= -17 \end{aligned}\right\}$$

Solving this (a and b) using (i) and (ii) gives $a = -3$, $b = 7$ then pluging into (iii) confirms that $\exists\ a, b$ s.t. u_4 is a combination of u_1 and u_2.

This leaves just u_5 to check but we know there is one vector left so our solution is $\boxed{\text{basis} = \{u_1, u_2, u_5\}.}$