# Selected Solutions to HW 7

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**#2:** Let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_2(a_1, a_2) = (2a_1 + 4a_2, -a_1 - a_2)$. Let $\beta = \{(1,2), (-1,1)\}$ and let $\gamma = \{(2,1), (2,0)\}$.

Compute $[T]_\beta^\gamma$.

First, pass $\beta$ through $T$.

$$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}10\\-3\end{pmatrix} \; ; \; T\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

now solve these two systems:

$$\begin{pmatrix}10\\-3\end{pmatrix} = \begin{pmatrix}2 & 2\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \Rightarrow \boxed{\begin{matrix}x_1 = -3\\x_2 = 8\end{matrix}}$$

$$\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}2 & 2\\1 & 0\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} \Rightarrow \boxed{\begin{matrix}y_1 = 0\\y_2 = 1\end{matrix}}$$

Vectors in the particular column space of $\gamma$.

$$[T]_\beta^\gamma = \begin{pmatrix}-3 & 0\\8 & 1\end{pmatrix}.$$

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**#3: § 2.2 Problem 10:** $\beta = \{v_1, v_2, \ldots, v_n\}$ $\beta$ an ordered basis. $v_0 = 0$.

$T(v_j) = v_j + v_{j-1}$ for $j = 1, \ldots, n$.

Compute $[T]_\beta$.

$$v_i = \begin{pmatrix}0\\0\\0\\1\\0\\\vdots\\0\end{pmatrix} \leftarrow i^{th} \text{ row}$$

$$[T]_\beta^\beta = \begin{bmatrix}1 & 1 & 0 & \cdots & \cdots & 0\\0 & 1 & 1 & & 0 & \vdots\\0 & 0 & 1 & \ddots & & \vdots\\\vdots & & 0 & \ddots & \ddots & \\ & & & & \ddots & 1\\0 & 0 & \cdots & \cdots & & 1\end{bmatrix}$$

# Sol'n HW 7:



**#4 §2.2 #13**: $V, W$ are V.S.s. Let $T, U$ be non-zero transformations: $V \to W$

If $R(T) \cap R(U) = \{0\}$ prove that $\{T, U\}$ is a linearly ind. subset of $\mathcal{L}(V, W)$.

- $R(T) \cap R(U) = \{0\}$ means that the transformations $T, U$ point all non-zero vectors in $V$ to a different set of vectors in $W$.

- Now the transformation $T$ and $U$ themselves are members of a space, $\mathcal{L}(V, W)$. So to check for independence we can use the same ideas as before. Namely, can one member of $\mathcal{L}(V,W)$ be made a multiple of another member of $\mathcal{L}(V,W)$. So is there a non-zero constant $c$ s.t. $T(x) = cU(x)$ for $x \in V$?

proof by contradiction. Assume that $\exists\, c \in \mathbb{R}$, $c \neq 0$, s.t. $T(x) = cU(x)$ for some $x \in V$ dependency
and $\exists\, y \in W, y \neq 0$ s.t. $T(x) = y$.

Notice that $y = \frac{1}{c} c y$ and that $U(x) = \frac{y}{c}$ so $y = c \cdot U(x)$ or $y = U(c \cdot x)$ and $cx \in V$ b/c $V$ is a vector space so $\underbrace{U(cx)}_{y} \in R(U) \Rightarrow R(T) \cap R(U) \neq \{0\}$.

This however is a contradiction, therefore $T$ and $U$ must be linearly independent in $\mathcal{L}(V, W)$. ∎

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**#7 §2.3 Problem 11**: Let $V$ be a vector space, let $T: V \to V$ be linear. Prove that $T^2 = T_0$ iff $R(T) \subseteq N(T)$. (Don't forget to prove both implications!)

($\Rightarrow$) Assume that $T^2 = T_0$. So $T(\underline{T(x)}) = \vec{0}\ \forall x \in V$.
$\Rightarrow \underline{T(x)} \in N(T)\ \forall x \in V$. Since this is for all $x \in V$
The set $T(x)$ is exactly the range, $R(T)$, as well
$\therefore T(x \in V) = R(T) \subseteq N(T)$.

($\Leftarrow$) Assume that $R(T) \subseteq N(T)$ then $T(x) = 0\ \forall x \in V$ then $T(T(x)) = 0\ \forall x \in V$ since $T(0) = 0$ always. $\therefore T^2 = T_0$ as desired.
$T^2 = T(T(x))$.

Q.E.D.