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These lecture notes are based on the textbook Linear Algebra, 4th edition, by Friedberg, Insel, and Spence, ISBN 0-13-008451-4. They are provided “as is” and as a courtesy only. They do not replace use of the textbook or attending class.
0 Foundational material: The appendices

0.1 Appendix A: Sets

Definition 0.1. A set is a collection of objects, called elements.

Example 0.2.
• \{1, 2, 3\} = \{2, 1, 1, 2, 3\} (no notion of “multiplicity”)
• \[1, 2\] = the interval of reals between 1 and 2, including 1 and 2.
• \(\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\) (later)
• \[
\begin{pmatrix}
8 \\
0 \\
-1
\end{pmatrix},
\begin{pmatrix}
1 \\
2 \\
2
\end{pmatrix}
\] = set of two vectors
• \(\emptyset\): the empty set

Given two sets \(A, B\), there are several operations that yield new sets from these. Most important are the following:

• \(A \cup B\) (union of \(A\) and \(B\))
• \(A \cap B\) (intersection of \(A\) and \(B\))
• \(A \times B = \{(a, b) : a \in A, b \in B\}\) (product of \(A\) and \(B\))

Definition 0.3. Let \(A\) be a set. A relation on \(A\) is a subset \(S\) of \(A \times A\). Write \(x \sim y\) if and only if \((x, y) \in S\).

Example 0.4. • \(A = \{1, 2, 3\}, S = \{(1, 2), (1, 3), (2, 3)\}\). This relation is “<”.
• \(A = \{1, 2, 3\}, S = \{(1, 2), (1, 3), (2, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}\). This relation is “\(\neq\)”.
• \(A = \{1, 2, 3\}, S = \{(1, 1), (2, 2), (3, 3)\}\). This relation is “=”.

Recall the following symbols. \(\forall\): “for all”, \(\exists\): “there exists”.

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Definition 0.5. Let $A$ be a set with a relation $S$. Then $S$ is called an equivalence relation if and only if

i. $\forall x \in A : x \sim x$ (reflexive)

ii. $\forall x, y \in A : x \sim y \Leftrightarrow y \sim x$ (symmetric)

iii. $\forall x, y, z \in A : (x \sim y \text{ and } y \sim z) \Rightarrow x \sim z$ (transitive)

Example 0.6. Let $A = \mathbb{Z}$. Let $x \sim y \Leftrightarrow \exists k \in \mathbb{Z} : x - y = 5k$. This defines an equivalence relation.

i. reflexive: Let $x \in \mathbb{Z}$. Then $x - x = 0 = 5 \cdot 0$. Done.

ii. symmetric: Let $x, y \in \mathbb{Z}$ with $x \sim y$. Then $x - y = 5k$ implies $y - x = 5 \cdot (-k)$. Done.

iii. transitive: Let $x, y, z \in \mathbb{Z}$ with $x \sim y$ and $y \sim z$. Then $x - y = 5k_1$ and $y - z = 5k_2$ implies (by adding the two equalities) $x - z = 5 \cdot (k_1 + k_2)$. Done.

0.2 Appendix B: Functions

Definition 0.7. Let $A, B$ be sets. A function $f : A \to B$ is a rule that associates to each element $x \in A$ a unique element of $B$, denoted $f(x)$. The set $A$ is called the domain, the set $B$ is called the codomain.

Definition 0.8. \begin{itemize}
  \item For $S \subseteq A$, $f(S) = \{ f(x) : x \in S \}$ (image of $S$ under $f$). $f(A)$ is called the range.
  \item For $T \subseteq B$, $f^{-1}(T) = \{ x \in A : f(x) \in T \}$ (pre-image of $T$ under $f$)
  \item $f : A \to B = g : A \to B \Leftrightarrow \forall x \in A : f(x) = g(x)$
\end{itemize}

Definition 0.9. \begin{itemize}
  \item $f : A \to B$ is injective if and only if $f(x) = f(y) \Rightarrow x = y$.
  \item $f : A \to B$ is surjective if and only if $\forall b \in B \exists a \in A : f(a) = b$.
  \item For $S \subseteq A$, the restriction of $f$ to $S$ is $f|_S : S \to B, x \mapsto f(x)$.
\end{itemize}
0.3 Appendix C: Fields

Definition 0.10. Let $A$ be a set. A binary operation is any map $A \times A \to A$. We are very familiar with $\mathbb{Q}$ and $\mathbb{R}$ and the properties that the two binary operations $+$ and $\cdot$ have.

Definition 0.11. A field $F$ is a set with two binary operations labelled $+$ and $\cdot$ such that

i. $a + b = b + a, \quad a \cdot b = b \cdot a$ (commutativity)

ii. $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity)

iii. $\exists 0 \in F : a + 0 = a$ $\forall a$

iv. $\exists 1 \in F : 1 \cdot a = a$ $\forall a$ (neutral elements)

v. $\forall a \in F : a \cdot 0 = 0$ $\forall a$

vi. $\forall a \in F : a \neq 0 \Rightarrow a \cdot 1 = a$ (inverse elements)

vii. $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributive law)

Theorem 0.12 (Cancellation Laws). Let $F$ be a field and $a, b, c \in F$.

i. $a + b = c + b \Rightarrow a = c$

ii. $a \cdot b = c \cdot b$ and $b \neq 0 \Rightarrow a = c$

Proof. Part i. Let $d$ be an additive inverse of $b$. Now, observe that $(a + b) + d = a$ and $(c + b) + d = c$. Done.

Part ii is done in detail in the textbook.

Proposition 0.13. The neutral element of addition is unique.

Proof. Let $0$ and $0'$ be two neutral elements of addition. Then

$0 = 0 + 0' = 0'$.

Example 0.14. Some examples of fields.

- $\mathbb{Q}, \mathbb{R}, \mathbb{C}$
\[ \mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3} | a, b \in \mathbb{Q}\} \] (on Homework 1).

\[ \mathbb{Z}/p\mathbb{Z} \] when \( p \) is a prime.

### 0.4 Appendix D: Complex Numbers

**Motivation:** In \( \mathbb{R} \), \( x^2 - 1 = 0 \) has two solutions, namely \(-1, 1\). However, the almost identical equation \( x^2 + 1 = 0 \) has no solutions. This means that the reals “leave something to be desired.” In response, we introduce the imaginary unit \( i \), which has the property \( i^2 = -1 \).

**Definition 0.15.** A *complex number* is an expression of the form \( z = a + bi \) with \( a, b \in \mathbb{R} \). Sum and product are defined by

\[
z + w = (a + bi) + (c + di) = a + c + (b + d)i
\]

and

\[
zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i.
\]

**Remark 0.16.** Memorize the multiplication by multiplying out as one would do naively, and then use \( i^2 = -1 \). Do some examples!

**Theorem 0.17.** The complex numbers with sum and multiplication as above form a field.

**Proof.** This just involves tedious checking of all the properties–you should try a few yourself at home. \( \square \)

**Remark 0.18.** The multiplicative inverse of \( z = a + bi \) is

\[
\frac{1}{a + bi} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2}.
\]

**Definition 0.19.** The complex conjugate of \( z = a + bi \) is \( \overline{z} = a - bi \).

**Proposition 0.20.**

i. \( \overline{\overline{z}} = z \)

ii. \( \overline{z + w} = \overline{z} + \overline{w} \)

iii. \( \overline{zw} = \overline{z} \cdot \overline{w} \)

iv. \( \overline{\frac{z}{w}} = \frac{\overline{z}}{\overline{w}} \)
Remark 0.21. It is now clear that there is a bijection $\mathbb{C} \to \mathbb{R}^2$ via $a + bi \mapsto (a, b)$. By Pythagoras’ Theorem, the length of a straight line from the origin to the point $(a, b)$ is $\sqrt{a^2 + b^2}$.

Definition 0.22. The absolute value (or modulus) of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

Remark 0.23. We have

$$z \bar{z} = (a + bi)(a - bi) = a^2 + b^2.$$ Thus,

$$|z| = \sqrt{z \bar{z}}.$$

Properties 0.24.

i. $|zw| = |z||w|$  
ii. $\frac{|z|}{|w|} = \frac{|z|}{|w|}$  
iii. $|z + w| \leq |z| + |w|$

Theorem 0.25 (Fundamental Theorem of Algebra). Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a complex polynomial (i.e., $a_i \in \mathbb{C}$). Then $\exists z_0 \in \mathbb{C} : p(z_0) = 0$.

Proof. No proof is given here. This is a comparatively hard theorem to prove.

1 Vector spaces

1.1 Introduction

Geometrically, a vector in, say, $\mathbb{R}^2$, is the datum of a direction and a magnitude. Thus, it can be represented by an arrow which points in the given direction and has the given length. Two vectors can be added using the parallelogram rule (see the textbook for some nice pictures explaining this).

Physically, the vectors may, e.g., represent forces that are exerted on an object. The result of the addition is the resulting net force that the object experiences when the original two forces are applied.

Algebraically, when $v = (a_1, a_2)$ and $w = (b_1, b_2)$, then $v + w = (a_1 + b_1, a_2 + b_2)$. Scalar multiplication is defined via $t(a_1, a_2) = (ta_1, ta_2)$. 

Definition 1.1. The vectors $v$ and $w$ are parallel if and only if $\exists t \in \mathbb{R}: tv = w$.

A vector can be interpreted as the displacement vector between its start and end point. If the start point is $(x_1, x_2)$ and the end point is $(y_1, y_2)$, then the displacement vector is $(y_1 - x_1, y_2 - x_2)$.

Definition 1.2. The line through the points $A = (x_1, x_2)$ and $B = (y_1, y_2)$ is

$$\{(x_1, x_2) + t(y_1 - x_1, y_2 - x_2) : t \in \mathbb{R}\}.$$  

Definition 1.3. The line through the points $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$ is

$$\{(x_1, x_2, x_3) + t(y_1 - x_1, y_2 - x_2, y_3 - x_3) : t \in \mathbb{R}\}.$$  

Definition 1.4. The plane through the points $A = (x_1, x_2, x_3)$, $B = (y_1, y_2, y_3)$ and $C = (z_1, z_2, z_3)$ (not all three on a line) is

$$\{(x_1, x_2, x_3) + s(y_1 - x_1, y_2 - x_2, y_3 - x_3) + t(z_1 - x_1, z_2 - x_2, z_3 - x_3) : s, t \in \mathbb{R}\}.$$  

Example 1.5. i. The line through $(1, 1, 2)$ and $(0, 3, -1)$ is

$$\{(1, 1, 2) + t(-1, 2 - 3) : t \in \mathbb{R}\}.$$  

ii. The plane through the points $A = (1, 0, -1)$, $B = (0, 1, 2)$ and $C = (1, 1, 0)$ is

$$\{(1, 0, -1) + s(-1, 1, 3) + t(0, 1, 1) : s, t \in \mathbb{R}\}.$$  

Now, observe that vector addition and scalar multiplication satisfy certain laws, e.g., $v + w = w + v$, $1 \cdot v = v$, $(ab)v = a(bv)$. Next, we will distill these obvious properties into an abstract definition.

## 1.2 Vector Spaces

Definition 1.6. A vector space (or linear space) $V$ over a field $F$ (think $F = \mathbb{R}$, or $\mathbb{C}$) is a set with a binary operation denoted “+” and a second map $\cdot : F \times V \to V$ such that

i. $\forall x, y \in V : x + y = y + x$
ii. \( \forall x, y, z \in V : (x + y) + z = x + (y + z) \)

iii. \( \exists 0 \in V : \forall x \in V : x + 0 = x \)

iv. \( \forall x \in V \exists y \in V : x + y = 0 \)

v. \( \forall x \in V : 1x = x \), where 1 is the neutral element of multiplication in \( F \)

vi. \( \forall a, b \in F \forall x \in V : (ab)x = a(bx) \)

vii. \( \forall a \in F \forall x, y \in V : a(x + y) = ax + ay \)

viii. \( \forall a, b \in F \forall x \in V : (a + b)x = ax + bx \)

**Definition 1.7.** The elements of \( F \) are called *scalars*. The elements of \( V \) are called *vectors*. Because of item ii above, sums like \( x + y + z + w \) are well-defined.

**Remark 1.8.** To simplify typing, we will usually not adorn vectors with an arrow, i.e., we will write \( x \) instead of \( \vec{x} \) and 0 instead of \( \vec{0} \). Note that the neutral element of addition in the field is also denoted with 0, but it should always be clear from the context what is meant.

**Quiz 1.**

1. (5 points) Let \( f : A \to B \) and \( g : B \to C \) be two functions. Assume that the composition \( g \circ f \) is injective. Does this necessarily imply that \( f \) is injective? Prove your answer.

2. (5 points) Let \( h : \mathbb{Z} \to \mathbb{Z} \) be defined by \( x \mapsto 2x + 4 \). Is \( h \) injective? Is \( h \) surjective? Prove your answers.

Answer to 1. Yes. Let \( x, y \in A \) with \( f(x) = f(y) \). Apply \( g \) to both sides of the equality. Then \( g(f(x)) = g(f(y)) \). Since \( g \circ f \) is injective, we can conclude \( x = y \), qed.

Answer to 2. Injective: Yes. Let \( x, y \in \mathbb{Z} \) with \( h(x) = h(y) \). Then \( 2x + 4 = 2y + 4 \). Subtracting 4 on both sides and then dividing by 2 yields \( x = y \), q.e.d. Surjective: No. For an arbitrary \( x \in \mathbb{Z} \), \( h(x) = 2x + 4 \) is clearly an even integer. Therefore the range of \( h \) cannot be \( \mathbb{Z} \), q.e.d.

**Example 1.9.**

i. (THE example, see later section on isomorphisms) Take a field \( F \). (We will mostly just take \( \mathbb{R} \), or perhaps \( \mathbb{C} \).) An \( n \)-tuple is \( (a_1, \ldots, a_n) \), where \( a_1, \ldots, a_n \in F \). Note that \( \{n\text{-tuples}\} \cong F^n \) naturally. Define \( (a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n) \). Also, \( c(a_1, \ldots, a_n) = (ca_1, \ldots, ca_n) \) for \( c \in F \).
ii. Mat_{m,n}(F) is a vector space with componentwise addition and scalar multiplication

The following examples of vector spaces are substantially different from the examples above. They are “infinite dimensional,” more about that later.

**Example 1.10.**

i. Let S be a set of real numbers Let \( F \) be the set of all real-valued functions on S. Then \( F \) is a vector space (over \( \mathbb{R} \)) with the usual addition and scalar multiplication of real-valued functions.

ii. Let S now be an interval of reals. Consider in \( F \) only those functions that are continuous. This is also a vector space (use the summation theorem for continuous functions from calculus)

iii. Consider in \( F \) only those functions that are differentiable. This is also a vector space (use the summation theorem for differentiable functions from calculus)

iv. Assume that \( x_0 \in S \). Then \( \{ f : S \to \mathbb{R} \mid f(x_0) = 0 \} \) is a vector space. (check it!). \( \{ f : S \to \mathbb{R} \mid f(x_0) = 1 \} \) is not!

Again, we would like to infer more properties of vector spaces from the original list of 8 properties. To start, we observe that the zero vector is unique, with the same proof as in the case of fields in the Introduction. Moreover, we also have a cancellation law:

**Theorem 1.11** (Cancellation law for vector spaces). Let \( V \) be a vector space and \( x, y, z \in V \). If \( x + z = y + z \), then \( x = y \).

**Proof.** Let \( v \) be such that \( z + v = 0 \) (condition iv). Then

\[
\begin{align*}
x & = x + 0 = x + (z + v) = (x + z) + v \\
& = (y + z) + v = y + (z + v) = y + 0 = y
\end{align*}
\]

due to conditions ii and iii.

**Corollary 1.12.** The additive inverse is unique.

**Theorem 1.13.** Let \( V \) be a vector space. Then the following statements are true.

i. \( \forall x \in V : 0x = \bar{0} \)
\[ \forall x \in V \forall a \in F : (\neg a)x = -(ax) = a(-x) \]

\[ \forall a \in F : a\vec{0} = \vec{0} \]

**Proof.** The textbook has detailed proofs of i and ii. The item iii is left to the reader. \qed

### 1.3 Subspaces

**Definition 1.14.** A subset \( W \) of a vector space \( V \) over the field \( F \) is called a *subspace of* \( V \) if \( W \) is a vector space with + and scalar multiplication from \( V \).

**Example 1.15.**
- \( \{\vec{0}\}, V \)
- \( \mathbb{R}^2 \cong \{(a, b, 0) | a, b \in \mathbb{R}\} \subset \mathbb{R}^3 \)
- The above examples 1.10ii and 1.10iii in 1.10i.

In order to verify that \( W \) is a subspace of \( V \), it is not necessary to check all the vector space axioms in the definition of a vector space. For example, the restricted addition is clearly commutative since it was already commutative before the restriction.

**Theorem 1.16.** A nonempty subset \( W \) of the vector space \( V \) is a subspace of \( V \) if and only if

i. \( \forall x, y \in W : x + y \in W \) (*closedness under +*)

ii. \( \forall c \in F \forall x \in W : c \cdot x \in W \) (*closedness under scalar multiplication*)

**Proof.** First, observe that the implication \( \Rightarrow \) is trivial. The proof of the other direction consists of some easy verifications. For example, let’s see why \( \vec{0} \in W \): Take an arbitrary element \( x \) of \( W \). Since \( W \) is nonempty, such an element exists. Now, simply observe that \( 0 \cdot x = \vec{0} \), which is an element of \( W \) by ii. The remaining details are left to the reader. \qed

More examples:
Example 1.17.  

• Let $W = \{(a, b)|a + b = 0\} \subset \mathbb{R}^2$. Closedness under $+$ is checked as follows. Let $(a, b), (c, d) \in W$. Then the result of the addition is $(a+c, b+d)$, which satisfies $(a+c) + (b+d) = (a+b) + (c+d) = 0 + 0 = 0$. Closedness under scalar multiplication is seen as follows. Let $(a, b) \in W$ and $c$ a scalar. Then the result of the scalar multiplication is $(ca, cb)$, which satisfies $ac + cb = c(a + b) = c0 = 0$.

• Let $W = \{(a, b, c)|3a - b + 2c = 0\} \subset \mathbb{R}^3$. Closedness under addition is checked as follows. Let $(a, b, c), (d, e, f) \in W$. Then the result of their addition is $(a+d, b+e, c+f)$, which satisfies $3(a+d) − b+e+2(c+f) = 3a−b+2c+3d−e+2f = 1+1 = 2 \neq 1$. Thus, $W$ is not closed under addition and not a subspace.

• Any intersection of subspaces in a vector space is itself a subspace.

A major class of examples is given by sums and direct sums of subspaces.

Definition 1.18. Let $V$ be a vector space. Let $S, T$ be nonempty subsets of $V$. Then let $S + T = \{x + y| x \in S, y \in T\}$. We call $S + T$ the sum of $S$ and $T$.

Definition 1.19. Let $V$ be a vector space. Let $W, U$ be subspaces of $V$. Then we call $V$ the direct sum of $W, U$ if $W + U = V$ and $W \cap U = \{0\}$. Write $V = W \oplus U$.

Proposition 1.20. Let $V$ be a vector space. Let $W, U$ be subspaces of $V$. Then the sum $W + U$ is a subspace of $V$ (containing both $W$ and $U$).

Proof. $(w_1 + u_1) + (w_2 + u_2) = (w_1 + w_2) + (u_1 + u_2)$, which is the sum of a vector in $W$, namely $w_1 + w_2$, and a vector in $U$, namely $u_1 + u_2$. Thus, $W + U$ is closed under addition. The closedness under scalar multiplication is completely analogous.  

Example 1.21.  

• $\{(a, b, 0, c)|a, b, c \in \mathbb{R}\} + \{(d, 0, e, f)|d, e, f \in \mathbb{R}\} = \mathbb{R}^4$. But this is not a direct sum.

• $\{(a, 0, 0, b)|a, b \in \mathbb{R}\} \oplus \{(0, c, d, 0)|c, d \in \mathbb{R}\} = \mathbb{R}^4$. This is a direct sum.
Quiz 2:

1. (5 points) Let $V$ be a vector space and let $W_1, W_2$ be subspaces of $V$. Prove that the intersection $W_1 \cap W_2$ is a subspace of $V$.

2. (5 points) Let $V$ be the vector space of $2 \times 2$ matrices with real entries. (You may assume without proof that $V$ is a vector space with the usual componentwise addition and scalar multiplication.) Recall that the determinant of such a matrix is defined to be
\[
\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc.
\]

Let $W$ be the subset of $V$ consisting of those matrices whose determinant is equal to 0. Is $W$ a subspace of $V$? Prove your answer.

Solution:

1. We need to prove that $W_1 \cap W_2$ is non-empty, closed under addition and closed under scalar multiplication.

   - Being subspaces, both $W_1$ and $W_2$ contain the zero-vector $\vec{0}$. So $\vec{0} \in W_1 \cap W_2 \neq \emptyset$.

   - Let $v, w \in W_1 \cap W_2$. In particular, $v, w \in W_1$ and by the closedness of $W_1$ under addition, $v + w \in W_1$. Also, $v, w \in W_2$ and by the closedness of $W_2$ under addition, $v + w \in W_2$. Altogether, $v + w \in W_1 \cap W_2$.

   - Let $v \in W_1 \cap W_2$ and $c \in F$. In particular, $v \in W_1$ and by the closedness of $W_1$ under scalar multiplication, $cv \in W_1$. Also, $v \in W_2$ and by the closedness of $W_2$ under scalar multiplication, $cv \in W_2$. Altogether, $cv \in W_1 \cap W_2$.

2. No! Simply observe that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in W$, but
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \notin W.
\]

This counterexample shows that $W$ is not closed under $+$. 

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1.4 Linear Combinations and systems of linear equations

Definition 1.22. Let $V$ be a vector space and $S$ a nonempty subset of $V$. We call $v \in V$ a linear combination of vectors in $S$ if there exist vectors $u_1, \ldots, u_n \in S$ and scalars $a_1, \ldots, a_n \in F$ such that $v = a_1 u_1 + \ldots + a_n u_n$.

Example 1.23. • $(3, 4, 1) = 3(1, 0, 0) + 4(0, 1, 0) + 1(0, 0, 1)$.

• If we want to write $(3, 1, 2)$ as a linear combination of $(1, 0, 1), (0, 1, 1), (1, 2, 1)$, how do we find the coefficients $a_1, a_2, a_3$? Answer: Make the Ansatz

$$(3, 1, 2) = a_1(1, 0, 1) + a_2(0, 1, 1) + a_3(1, 2, 1)$$

and solve the system of linear equations

$$a_1 + a_3 = 3, \quad a_2 + 2a_3 = 1, \quad a_1 + a_2 + a_3 = 2.$$  

Solution: $a_1 = 2, a_2 = -1, a_3 = 1$.

• Find the $a_i$ in a given situation may or may not be possible. E.g., writing

$$(3, 1, 2) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(1, 2, 0)$$

is clearly impossible.

• There may be many choices for the $a_i$:

$$(2, 6, 8) = a_1(1, 2, 1) + a_2(-2, -4, -2) + a_3(0, 2, 3) + a_4(2, 0, -3) + a_5(-3, 8, 16)$$

is equivalent to

$$(a_1, a_2, a_3, a_4, a_5) \in \{(−4 + 2s − t, s, 7 − 3t, 3 + 2t, t)|s, t \in \mathbb{R}\}.$$  

(There are two “free variables”.)

There are three types of operations that we used to solve the above systems of linear equations:

i. Interchange the order of any two equations.

ii. Multiply an equation by a nonzero scalar.

iii. Add one equation to another.
Key point: These operations do not change the set of solutions.

**Definition 1.24.** Let $V$ be a vector space. Let $S$ be a nonempty subset of $V$. We call $\text{span}(S)$ the set of all vectors in $V$ that can be written as a linear combination of vectors in $S$.

**Example 1.25.** Let $S = \{(1,0,0),(0,1,0),(2,1,0)\}$. Then $\text{span}(S) = \{(s,t,0) | s,t \in \mathbb{R}\}$.

**Theorem 1.26.** The span of any subset $S$ of a vector space $V$ is a subspace of $V$.

**Proof.** Let $v = a_1u_1 + \ldots + a_nu_n \in \text{span}(S)$. Then $cv = (ca_1)u_1 + \ldots + (ca_n)u_n \in \text{span}(S)$. Thus, closedness under scalar multiplication is ok.

Let $v = a_1v_1 + \ldots + a_nv_n \in \text{span}(S)$ and let $w = b_1w_1 + \ldots + b_mw_m \in \text{span}(S)$. Then

$$v + w = a_1v_1 + \ldots + a_nv_n + b_1w_1 + \ldots + b_mw_m.$$ 

Thus, closedness under addition is ok.

**Example 1.27.**

- $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ spans $\mathbb{R}^3$.
- $S = \{(1,2),(2,1)\}$ spans $\mathbb{R}^2$.
- $S = \{(1,1),(2,2)\}$. $\text{span}(S) = \{(s,s) | s \in \mathbb{R}\} \neq \mathbb{R}^2$.
- $S = \{(1,2)\}$ does not span $\mathbb{R}^2$.
- Which $(a,b,c)$ are in $\text{span}(\{(1,1,2),(0,1,1),(2,1,3)\})$? Answer: Those that satisfy $a + b = c$.

### 1.5 Linear dependence and linear independence

Motivation: Let $W$ be a subspace of $V$. We are interested in a set $S \subset W$ such that $\text{span}(S) = W$ and $S$ is “as small as possible”.

**Definition 1.28.** A subset $S$ of a vector space $V$ is called *linearly dependent* if there exist a finite number of vectors $u_1, \ldots, u_n \in S$ and scalars $a_1, \ldots, a_n$, not all equal to zero, such that

$$a_1u_1 + \ldots + a_nu_n = 0.$$ 

We also say that the vectors in $S$ are linearly dependent.
Quiz 3: (10 points) Find the condition(s) for \((a, b, c, d) \in \mathbb{R}^4\) to be in \(\text{span}\{(1, 0, 1, -1), (-1, -2, -1, -1), (0, 1, 0, 1), (1, 3, 1, 2)\}\).

Solution:

Make the Ansatz

\[
a_1(1, 0, -1, 1) + a_2(-1, -2, -1, -1) + a_3(0, 1, 0, 1) + a_4(1, 3, 1, 2) = (a, b, c, d).
\]

We have to determine for which values of \(a, b, c, d\) the above system is consistent, i.e., solvable. Reducing this system to echelon form yields:

\[
\begin{align*}
a_1 - a_2 + a_4 &= a \\
-2a_2 + a_3 + 3a_4 &= b \\
0 &= c - a \\
0 &= d + a - b
\end{align*}
\]

It is now clear that the system is consistent if and only if \(c = a\) and \(b = a + d\).

Done.

Example 1.29. Let \(S = \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}\). Then

\[
4(1, 3, -4, 2) - 3(2, 2, -4, 0) + 2(1, -3, 2, -4) + 0(-1, 0, 1, 0) = (0, 0, 0, 0).
\]

Thus, \(S\) is linearly dependent.

Definition 1.30. If \(S\) is not linearly dependent, we say \(S\) is linearly independent.

Remark 1.31. Linear independence is equivalent to: \(\sum a_i v_i = \vec{0} \Rightarrow \text{all } a_i = 0\).

Remark 1.32. The empty set \(\emptyset\) is linearly independent. The singleton set \(\{v\}\) is linearly independent if and only if \(v \neq \vec{0}\).

Theorem 1.33. Let \(V\) be a vector space. If \(S_1 \subseteq S_2\) and \(S_1\) is linearly dependent, then \(S_2\) is linearly dependent.

Proof. This is immediate from the definition.

Theorem 1.34. Let \(S\) be a linearly independent subset of \(V\). Let \(v \in V \setminus S\). Then \(S \cup \{v\}\) is linearly dependent if and only if \(v \in \text{span}(S)\).
Proof. “⇒”. Write \( a_1 u_1 + \ldots + a_n u_n + a_{n+1} v = 0 \) with not all \( a_i \) equal to zero and \( u_i \in S \).

Claim: \( a_{n+1} \neq 0 \).

Proof of claim: If \( a_{n+1} = 0 \), then at least one of \( a_1, \ldots, a_n \) is not equal to zero and \( a_1 u_1 + \ldots + a_n u_n = 0 \). Contradiction to linear independence of \( S \).

So, \( a_{n+1} \neq 0 \), and we can write \( v = \frac{-a_1}{a_{n+1}} u_1 + \ldots + \frac{-a_n}{a_{n+1}} u_n \), qed.

“⇐”. Write \( v = a_1 u_1 + \ldots + a_n u_n \). Then \( a_1 u_1 + \ldots + a_n u_n + (-1)v = 0 \).

Thus, \( S \cup \{v\} \) is linearly dependent, qed. \( \Box \)