§1.4 (b) and (c). Matrices haven’t yet been introduced so we’ll solve these by typical algebraic manipulations.

(b) \[3x_1 - 7x_2 + 4x_3 = 10 \quad (i) \]
\[x_1 - 2x_2 + x_3 = 3 \quad (ii)\]
\[2x_1 - x_2 - 2x_3 = 0 \quad (iii)\]

Let’s create new equations by combining multiples of (i), (ii), (iii).

-2 \,(ii) + \,(iii) \quad 3x_2 - 4x_3 = 0 \quad \Rightarrow \quad x_2 = \frac{4}{3}x_3 ; \text{ plug this into other new equation.}

-3 \,(ii) + \,(i) \quad -x_2 + x_3 = 1

\[\frac{-4}{3}x_3 + x_3 = 1\]
\[\frac{-1}{3}x_3 = 1\]
\[x_3 = -3\]
\[\Rightarrow \quad x_2 = -4\]

Thus \[x_2 = -4\] to find \[x_1\]

\[x_1 = 3 - 8 + 3\]
\[= -2\]

\[\boxed{\text{Solution} = \begin{cases} x_1 = -2 \\ x_2 = -4 \\ x_3 = -3 \end{cases}}\]

(c) \[x_1 + 2x_2 - x_3 + x_4 = 5 \quad (i)\]
\[x_1 + 4x_2 - 3x_3 - 3x_4 = 6 \quad (ii)\]
\[2x_1 + 3x_2 - x_3 + 4x_4 = 8 \quad (iii)\]

\[\text{Note:} \quad (ii) - (i): = 2x_2 - 2x_3 - 4x_4 = 1 \quad (iv)\]
\[\text{Note:} \quad (iii) - 2(i): = -x_2 + x_3 + 2x_4 = -3 \quad (v)\]

\[\text{Note:} \quad (iv) + 2(v): = 0 = -3, \quad \text{but} \ 0 \neq -3 \quad \text{so this is a contradiction so this system of equations has NO solutions.}\]
§1.4 #3(c) : Determine whether the first vector in the set of vectors can be written as a combination of the other two.

\{ (3, 4, 1), (1, -2, 1), (-2, -1, 1) \}

This question is very similar to #2. We just need to know how to put, or represent, this set of vectors as a system of equations.

\[(3, 4, 1) = a(1, -2, 1) + b(-2, -1, 1)\] can be written as

\[\begin{align*}
(i) & \quad a - 2b = 3 \\
(ii) & \quad -2a - b = 4 \\
(iii) & \quad a + b = 1
\end{align*}\]

(Note: If this system has solutions then one equation needs to be a linear combination of the other two.)

\[2(i) + (ii) : -5b = 10 \Rightarrow b = -2 \quad \text{then by (iii)} \quad a = -1\]

So now let's make sure that these values satisfy (iii). If they do then this system is consistent and \(\{a = -1, \ b = -2\}\) is indeed the solution.

Plug \(a = -1, \ b = -2\) into (iii)

\[-1 - 2 = -3 \neq 1 \Rightarrow (3, 4, 1) \text{ cannot be expressed as a linear combination of } (1, -2, 1) \text{ and } (-2, -1, 1).\]
Find a subset of $S$ that is a basis for $\mathbb{R}^3$.

All we must do to find the basis is isolate the three vectors in $S$ which are independent.

- We can immediately see that $-4u_1 = u_3$. So throw out $u_3$.
- We see that $u_1$ and $u_2$ are independent. Now we must check whether $u_4$ or $u_5$ is the final vector composing the basis for $\mathbb{R}^3$.
- We can use the same techniques employed in the previous questions. Let's check $u_4$.

(i) \[ 2a + 6 = 1 \]
(ii) \[ -3a + 4b = 37 \]
(iii) \[ a - 2b = -17 \]

Solving this for $(a, b)$ using (i) and (ii) gives $a = -3$, $b = 7$.

This plugging into (iii) confirms that $\exists a, b \text{ s.t. } u_4$ is a combination of $u_1$ and $u_2$.

This leaves just $u_5$ to check but we know there is one vector left so our solution is $\text{basis} = \{u_1, u_2, u_5\}$.