§1.6 #14: Find bases for the following subspaces of $F^5$

$W_1 = \{ (a_1,a_2,a_3,a_4,a_5) \in F^5 : a_1 - a_3 - a_4 = 0 \}$

First, recognize which values are free to take any number in $F$ without affecting $a_1 - a_3 - a_4 = 0$. We see that $a_2$ and $a_5$ are not present, so

$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ are vectors in the basis for $F^5$.

For the remaining vectors we must construct the basis vector in a way that satisfies the given equation.

- allowing $a_3$ to be zero $\Rightarrow$ $a_1 = a_4 = 7 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is a basis vector.
- allowing $a_1$ to be zero $\Rightarrow$ $a_3 = -a_4 = 7 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ is a basis vector.

We stop here b/c if we carry this scheme forward and allow $a_4 = 0$ then a basis vector could be $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ but this is a linear combination of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$, so it must be scrapped. $\therefore \text{Basis}(W_1) = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \}$

and $\dim(W_1) = 4$. 
Using the same thought process as before, we have that a basis for \( W_2 \) is \(
\begin{pmatrix}
10
00
00

11
10
10

1
1

-1
0
0
\end{pmatrix}
\) with \( \dim W_2 = 2 \)

\#5: Let \( G = \{(1, -1, 0, 1), (1, 0, 1, 0), (1, 2, 2, 0), (1, 2, 2, 0)\} \)

Let \( L = \{(-1, 4, 2, 0)\} \)

(a) Show that \( G \) spans \( \mathbb{F}^4 \). Want to show \( \begin{pmatrix}
1 & 1 & 0 & \alpha_1
0 & 1 & 2 & 0
0 & 1 & 2 & 0
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha_1
\alpha_2
\alpha_3
\alpha_4
\end{pmatrix}
= \begin{pmatrix}
\alpha_1
\alpha_2
\alpha_3
\alpha_4
\end{pmatrix}
\)

reduces to a matrix with full rank.

We observe that this matrix is full rank \( \iff \) each column has a pivot.
The right hand side are numbers in the underlying field undergoing operations that put them back into the field (addition and scalar multiplication).

(b) Choose \( H = \{(1, -1, 0, 1), (1, 0, 1, 0), (1, 2, 2, 0)\} \); \( L = \{(-1, 4, 2, 0)\} \).

reduce HUL to see if all vectors are linearly independent.

\( \begin{pmatrix}
1 & 1 & -1 & \alpha_1
0 & 1 & 2 & 0
0 & 1 & 2 & 0
0 & 0 & 1 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0
0 & 1 & 2 & 0
0 & 0 & 1 & 0
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha_1
\alpha_2
\alpha_3
\alpha_4
\end{pmatrix}
\)

By the Replacement Theorem

\( \text{HUL generates } V \)

\( \text{Span } G = \text{Span } (\text{HUL}) \)

Again, we're reduced to a full rank matrix, \( \therefore \) independence of vectors.
1.6 # 29 Proof: If $W_1$ and $W_2$ are finite dimensional subspaces of $V$, where $V$ is a vector space, then $W_1 + W_2$ is finite dimensional and that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

(a) Let $\text{Basis}_3(W_1 \cap W_2) = \{u_1, u_2, \ldots, u_k\}$ then $\dim(W_1 \cap W_2) = k$.

(b) Let $\text{Basis}_3(W_1) = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m\}$ then $\dim(W_1) = k + m$.

(c) Let $\text{Basis}_3(W_2) = \{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_p\}$ then $\dim(W_2) = k + p$.

Let $\text{Basis}_3(W_1 + W_2) = \{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_p\}$ then $\dim(W_1 + W_2) = (k + m) + (k + p) - k = k + m + p$.

Note: $W_1 \subseteq W_1 \cap W_2$,

$W_2 \subseteq W_1 \cap W_2$.

$\{v_1, \ldots, v_m\} \subseteq W_1 \cap W_2$.

$\{w_1, \ldots, w_p\} \subseteq W_2 \cap W_1$.

(b) $W_1$ and $W_2$ finite dimensional subspaces and $V = W_1 + W_2$.

Prove $V = W_1 \oplus W_2$ iff $\dim(V) = \dim(W_1) + \dim(W_2)$.

($\Rightarrow$) Assume $V = W_1 \oplus W_2$.

$W_1 \cap W_2 = \{\emptyset\}$ by assumption.

Since $V = W_1 + W_2$ is given and by above we have $\dim(V) = \dim(W_1) + \dim(W_2)$.

($\Leftarrow$) Assume $\dim(V) = \dim(W_1) + \dim(W_2)$.

By above part (a), we have $\dim(W_1 \cap W_2) = \emptyset$ and $V = W_1 + W_2$ as given.

These together define the direct sum, so $V = W_1 \oplus W_2$. 

\[ \square \]