Section 2.4 Problem 4: Let $A, B \in M_{mn}$ and be invertible matrices. Prove that $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: $A, B$ are given as invertible so $AA^{-1} = I_n = A^{-1}A$ and $BB^{-1} = I_n = B^{-1}B$.

If $AB$ is invertible then $AB(AB)^{-1} = I_n$ should be true and $(AB)^{-1}AB = I_n$ should be true.

1. $AB(AB)^{-1} = I_n$; multiply by $B^{-1}A^{-1}$ on the left.
   
   $(B^{-1}A^{-1})AB = B^{-1}A^{-1}$; use $A^{-1}A = B^{-1}B = I_n$

   $\therefore (AB)^{-1} = B^{-1}A^{-1}$

2. $(AB)^{-1}AB = I_n$; multiply by $B^{-1}A^{-1}$ on the right.

   $(AB)^{-1}ABB^{-1}A^{-1} = B^{-1}A^{-1}$; use $BB^{-1} = AA^{-1} = I_n$

   $\therefore (AB)^{-1} = B^{-1}A^{-1}$

See definition on pg 100. $AB$ is invertible b/c $\exists X$ s.t. $ABX = I$ and $XAB = I$. $X = B^{-1}A^{-1}$ and $B^{-1}A^{-1} = (AB)^{-1}$. $\Box$
5.2.4 Problem 7: Let $A$ be an $(m \times n)$ matrix.

(a) Suppose that $A^2 = 0$. Prove that $A$ is not invertible.

(b) Suppose that $AB = 0$ for some nonzero $m \times n$ matrix $B$. Could $A$ be invertible? Explain.

(a) Proof by Contradiction:

Suppose that $A$ is invertible. Then $\exists \, B \in M_{mn}$ s.t. $AB = I$. Recall $A^2 = 0 \Rightarrow A^2B = 0$

$A^3B = A(AB) = 0 \Rightarrow A^2I = 0 \Rightarrow A = 0$ which contradicts that $A$ is invertible.

$\therefore$ $A$ is not invertible.

(b) Proof by Contradiction:

Suppose that $A$ is invertible. Then $B = B$ can be seen as $B = (A^{-1}A)B = A^{-1}(AB)$.

But $AB = 0$ by assumption so $B = A^{-1}(0) = B = 0$ which is a contradiction.

$\therefore$ $A$ cannot be invertible.

4.3.2.4 Problem 10: Let $B$ be an $(n \times n)$ and invertible. Define $\Phi : M_{nn}(F) \rightarrow M_{nn}(F)$ by 

$\Phi(A) = B^{-1}AB$. Prove that $\Phi$ is an isomorphism.

Need to show two things about $\Phi$: 1. that it's linear 2. that it's invertible.

1. $\Phi(aA_1 + aA_2) = B^{-1}(aA_1 + aA_2)B = B^{-1}(aA_1)B + B^{-1}(aA_2)B$

$\Phi(aA_1 + aA_2) = a_1B^{-1}A_1B + a_2B^{-1}A_2B = a_1\Phi(A_1) + a_2\Phi(A_2)$. $\therefore \Phi$ is linear.

2. If $\Phi$ is invertible then $\exists \Omega : M_{nn}(F) \rightarrow M_{nn}(F)$ and $\Phi(\Omega(x)) = x$ and $\Omega(\Phi(y)) = y$

Let $\Omega(A) = BAB^{-1}$. $\Phi(\Omega(x)) = \Phi(BXB^{-1}) = B^{-1}(BXB^{-1})B = (B^{-1}B)X(B^{-1}B) = X$.

$\Omega(\Phi(y)) = \Omega(B^{-1}YB) = BB^{-1}YBB^{-1} = (BB^{-1})Y(BB^{-1}) = Y$. $\therefore \Phi$ invertible. $\therefore \Phi$ is an Isomorphism.