UH - Math 4377/6308 - Dr. Heier - Spring 2010

Midterm Exam

Thursday, March 11, 2010

Solution

Print your **NAME**:

Solve all of the five problems. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit. This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. The time allowed will be announced by the proctor.

Problem 1 ____________________/20 points

Problem 2 ____________________/20 points

Problem 3 ____________________/20 points

Problem 4 ____________________/20 points

Problem 5 ____________________/20 points

Total ____________________/100 points

1a. (10 points) Let $x, y \in \mathbb{R}$. Let $x \sim y$ if and only if there exists an integer n such that $y = 2^n x$. Prove that $\sim$ is an equivalence relation. (Hint: $2^0 = 1$.)

1b.

(i) (5 points) Solve $z^2 - 2z + 2 = 0$ in $\mathbb{C}$ (i.e., the field of complex numbers).

(ii) (5 points) For an arbitrary non-zero complex number of the form $z = a + bi$, give the multiplicative inverse z^{-1} in the form $c + di$. Justify your answer carefully.

1a. Reflexive: $x \sim x \Leftrightarrow \exists n \in \mathbb{Z}: x = 2^n x$

Take $n = 0$. Then $x = 2^0 x = 1 \cdot x = x$ ✓

Symm.: $x \sim y \Rightarrow \exists n \in \mathbb{Z}: y = 2^n x \Leftrightarrow \exists n \in \mathbb{Z}: 2^{-n} y = x$

Take $m = -n$. Then $x = 2^m y$ ✓

Trans: $x \sim y \wedge y \sim z \Rightarrow y = 2^{n_1} x \wedge \underline{z} = 2^{n_2} y = 2^{n_2} 2^{n_1} x$

$= 2^{n_2 + n_1} \underline{x}$

Take $m = n_1 + n_2 \Rightarrow z = 2^m x$. ✓

1b. i) $z^2 - 2z + 2 = 0$

$$z = 1 \pm \sqrt{1 - 2} = 1 \pm \sqrt{-1} = 1 \pm i$$

ii) $$\frac{1}{a+bi} = \frac{a - bi}{(a+bi)(a-bi)} = \frac{a - bi}{a^2 - (bi)^2} =$$

$$= \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2} i$$

2a. Determine if the following subsets of the vector space of 2×2 matrices with real entries are subspaces. You may assume as true that the set of 2×2 matrices with real entries forms a vector space with the usual addition and scalar multiplication. Justify your answer carefully.

(i) (7 points) $\left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_3 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}$

(ii) (7 points) $\left\{ \begin{pmatrix} a_1 & a_2^2 \\ a_2 & a_3 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}$

2b. (6 points) Assume as given that the $n \times n$ matrices with real entries form a vector space. Is the subset of invertible matrices a linear subspace of this vector space? Justify your answer carefully.

2a i) $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_3 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_3 \end{pmatrix} = \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 = & a_3+b_3 \end{pmatrix}$ ✓

$c \begin{pmatrix} a_1 & a_2 \\ a_3 & a_3 \end{pmatrix} = \begin{pmatrix} ca_1 & ca_2 \\ ca_3 = & ca_3 \end{pmatrix}$ ✓ YES.

ii) $3 \cdot \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{in set}} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \to$ not in set.

No.

2b) For an invertible A, $\overset{\text{scalar}}{0} \cdot A = 0$-matrix, so <u>not</u> closed.

3a. (10 points) Find a basis for the following subspace W of $\mathbb{R}^5$:

$W = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 : a_1+a_2+a_3+a_4+a_5 = 0, a_1 = a_2+3a_3, a_1+a_4 = 0\}$.

3b. (10 points) Find nullity and rank of

$T : \mathbb{R}^5 \to \mathbb{R}^3, (a_1, a_2, a_3, a_4, a_5) \mapsto (a_1+a_2+a_3, -a_1+2a_2+a_4+a_5, 3a_2+a_3+a_4+a_5)$.

3a.
$$\left.\begin{array}{l} a_1 + a_4 = 0 \\ a_1 - a_2 - 3a_3 = 0 \\ a_1 + a_2 + a_3 + a_4 + a_5 = 0 \end{array}\right\} \quad \left.\begin{array}{l} a_1 + a_4 = 0 \\ -a_2 - 3a_3 - a_4 = 0 \\ a_2 + a_3 + a_5 = 0 \end{array}\right\}$$

$$a_1 + a_4 = 0$$
$$-a_2 - 3a_3 - a_4 = 0$$
$$-2a_3 - a_4 + a_5 = 0 \rightsquigarrow a_3 = \frac{1}{2}a_5 - \frac{1}{2}a_4$$
$$\rightsquigarrow a_2 = -3a_3 - a_4 = -\frac{3}{2}a_5 + \frac{1}{2}a_4$$
$$a_1 = -a_4$$

$$W = \left\{\left(-a_4, -\frac{3}{2}a_5 + \frac{1}{2}a_4, \frac{1}{2}a_5 - \frac{1}{2}a_4, a_4, a_5\right) \middle| a_4, a_5 \in \mathbb{R}\right\}$$
$$= \left\{a_4\left(-1, \frac{1}{2}, -\frac{1}{2}, 1, 0\right) + a_5\left(0, -\frac{3}{2}, \frac{1}{2}, 0, 1\right) \middle| a_4, a_5 \in \mathbb{R}\right\}$$
$$\rightsquigarrow \mathcal{B} = \left\{\left(-1, \frac{1}{2}, -\frac{1}{2}, 1, 0\right), \left(0, -\frac{3}{2}, \frac{1}{2}, 0, 1\right)\right\}$$

3b. Find nullity:

$$\left.\begin{array}{l} a_1 + a_2 + a_3 = 0 \\ -a_1 + 2a_2 + a_4 + a_5 = 0 \\ 3a_2 + a_3 + a_4 + a_5 = 0 \end{array}\right\} \quad \underbrace{\begin{array}{l} a_1 + a_2 + a_3 = 0 \\ 3a_2 + a_3 + a_4 + a_5 = 0 \end{array}}_{a_3,\, a_4,\, a_5 \text{ free variables}}$$

$\Rightarrow$ nullity $= \underline{\underline{3}}$

Find rank: nullity + rank = dim $\mathbb{R}^5$

$\Rightarrow$ rank $= 5 - 3 = \underline{2}$

4a. Let $G = \{(-1,0,-1),(3,2,1),(1,2,1)\}$. Let $L = \{(1,1,0),(2,1,1\}$.

(i) (7 points) Show that G spans $\mathbb{R}^3$. (Since it has 3 elements, G is then automatically a basis, but we are only interested in the spanning property.)

(ii) (7 points) Find a vector $v \in G$ such that $\{v\} \cup L$ spans $\mathbb{R}^3$. Prove the spanning property with an explicit computation.

4b. (6 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(a_1, a_2) = (a_1 + a_2, a_1 - a_2)$. Let $\beta = \{(1,1),(-2,1)\}$ and $\gamma = \{(1,0),(1,-1)\}$. Compute $[T]_\beta^\gamma$.

4a) (i)

$$\left.\begin{matrix} -1 & 3 & 1 \\ 0 & 2 & 2 \\ -1 & 1 & 1 \end{matrix}\right\} \quad \left.\begin{matrix} -1 & 3 & 1 \\ 0 & 2 & 2 \\ 0 & -2 & 0 \end{matrix}\right\} \quad \left.\begin{matrix} -1 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 2 & 2 \end{matrix}\right\}$$

$$\left.\begin{matrix} -1 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{matrix}\right\} \text{ Echelon Form} \Rightarrow \text{spanning OK}$$

4a)(ii)

$$a_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \left.\begin{aligned} a_1 + 2a_2 - a_3 &= 0 \\ -a_2 + a_3 &= 0 \\ a_2 - a_3 &= 0 \end{aligned}\right\} \Rightarrow \text{lin. dep.} \quad \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \text{ not usable}$$

$$a_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \left.\begin{aligned} a_1 + 2a_2 + 3a_3 &= 0 \\ a_1 + a_2 + 2a_3 &= 0 \\ a_2 + a_3 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} a_1 + 2a_2 + 3a_3 &= 0 \\ -a_2 - a_3 &= 0 \\ a_2 + a_3 &= 0 \end{aligned}\right\} \Rightarrow \text{lin. dep} \quad \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \text{ not usable}$$

$$a_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \left.\begin{aligned} a_1 + 2a_2 + a_3 &= 0 \\ a_1 + a_2 + 2a_3 &= 0 \\ a_2 + a_3 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} a_1 + 2a_2 + a_3 &= 0 \\ -a_2 + a_3 &= 0 \\ a_2 + a_3 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} a_1 + 2a_2 + a_3 &= 0 \\ -a_2 + a_3 &= 0 \\ 2a_3 &= 0 \end{aligned}\right\} \Rightarrow \text{lin. indep.} \checkmark$$

4b. $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} = 2\cdot\begin{pmatrix}1\\0\end{pmatrix} + 0\cdot\begin{pmatrix}1\\-1\end{pmatrix}$

$T\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}-1\\-3\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}1\\-1\end{pmatrix}$

$\Leftrightarrow$ $b = 3$ and $a = -4$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix}2 & -4\\0 & 3\end{pmatrix}$

5a. (10 points) Let V be a vector space over the real numbers. Let $\{v_1, v_2\}$ be a basis for V. Is $\{v_1 + v_2, v_2\}$ also a basis for V? Prove your answer.

5b. (10 points) Let V be a finite-dimensional vector space over the real numbers. Let $T : V \to V$ be a linear transformation. Prove that **if** T is onto **then** T carries linearly independent subsets of V onto linearly independent subsets of V.

5a. $\{v_1 + v_2, v_2\}$ contains $2 = \dim V$ vectors.

Suffices to check lin. indep.

$$a(v_1 + v_2) + b v_2 = \vec{0}$$

$$\Leftrightarrow a v_1 + (a+b) v_2 = 0$$

$\underset{v_1, v_2 \text{ lin. indep}}{\Rightarrow}$ $a = 0$ and $a + b = 0$ $\Rightarrow$ $\underline{a = 0 = b}$.

5b. Since T is onto, nullity $= \dim V - \operatorname{rank}(T) = 0$

$\Rightarrow T$ is 1-1.

For $\{v_1, .., v_k\}$ lin. indep, write

$$a_1 T(v_1) + \ldots + a_k T(v_k) = \vec{0}$$

$\underset{T \text{ linear}}{\Rightarrow}$ $T(a_1 v_1 + \ldots + a_k v_k) = \vec{0}$

$\underset{T \text{ 1-1}}{\Rightarrow}$ $a_1 v_1 + \ldots + a_k v_k = \vec{0}$

$\underset{\{v_1, .., v_k\} \text{ lin. indep.}}{\Rightarrow}$ $a_1 = \ldots = a_k = 0$ $\square$