1. (1 point) Give an example of a topological space which has all the properties of a topological manifold except that it is not Hausdorff. Discuss your example in detail.

2. (1 point) Give an example of a topological space which has all the properties of a topological manifold except that it is not second countable. Discuss your example in detail.

3. (1 point) Prove that \( \mathbb{P}^n \) is Hausdorff and second countable.

4. (1 point) Consider the real line \( \mathbb{R} \) as a topological manifold in the usual sense. Prove that there exist uncountably many distinct smooth structures on \( \mathbb{R} \).

5. (1 point) Let \( M \) be a smooth manifold. Let \( f : M \to \mathbb{R}^k \) be a smooth function as defined in class. Prove that \( f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k \) is smooth for every chart \( (U, \varphi) \) in the maximal atlas defining \( M \).

6. (1 point) Let \( M \) be a smooth manifold of dimensional at least 1. Prove that \( C^\infty \) is an infinite dimensional vector space.

7. (1 point) Let \( P : \mathbb{R}^{n+1} \to \mathbb{R}^{k+1} \) be a smooth map, and suppose that for some \( d \in \mathbb{Z} \), \( P(\lambda x) = \lambda^d P(x) \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( x \in \mathbb{R}^{n+1} \). (Such a map is said to be homogenous of degree \( d \).) Prove that the map \( \tilde{P} : \mathbb{P}^n \to \mathbb{P}^k \) defined by \( \tilde{P}([x]) := [P(x)] \) is well-defined and smooth.

8. (1 point) Let \( G \) be a smooth manifold with a group structure such that the map \( G \times G \to G \) given by \( (g, h) \mapsto gh^{-1} \) is smooth. Prove that \( G \) is a Lie group.

9. (1 point) Let \( M \) be a topological space with the property that for every open cover \( \mathcal{X} \) of \( M \), there exists a partition of unity subordinate to \( \mathcal{X} \). Prove that \( \mathcal{X} \) is paracompact.

10. Let \( M \) be a smooth manifold, let \( B \subset M \) be a closed subset, and let \( \delta : M \to \mathbb{R} \) be a positive continuous function.

   (a) (0.5 points) Using a partition of unity, prove that there is a smooth function \( \tilde{\delta} : M \to \mathbb{R} \) such that \( 0 < \tilde{\delta}(x) < \delta(x) \) for all \( x \in M \).

   (b) (0.5 points) Prove that there is a continuous function \( \psi : M \to \mathbb{R} \) that is smooth and positive on \( M \setminus B \), identically equal to zero on \( B \), and satisfies \( \psi(x) < \delta(x) \) everywhere on \( M \). Hint: Consider \( 1/(1 + f) \), where \( f : M \setminus B \to \mathbb{R} \) is a positive exhaustion function.