UH - Math 3330 - Dr. Heier - Midterm Exam - Spring 2014

Thursday, March 20, 2014

Print your NAME: Solution

Solve all of the five problems. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit. This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. The time allowed will be announced by the proctor.

Problem 1 ______________________________ /20 points

Problem 2 ______________________________ /20 points

Problem 3 ______________________________ /20 points

Problem 4 ______________________________ /20 points

Problem 5 ______________________________ /20 points

Total ______________________________ /100 points

1. (a) (10 points) For sets A and B, prove the following equivalence. (Hint: Recall that $A \subset B$ means $x \in A \Rightarrow x \in B$.)

$$A \subset B \text{ if and only if } A \cup B = B.$$

(b) (10 points) For $x, y \in \mathbb{Z}$, let $x \sim y$ if and only if $x + y$ is even. Prove that $\sim$ is an equivalence relation.

a) "$\Rightarrow$" First, we prove $A \cup B \subset B$.

let $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$.

$A \subset B$

Second, $B \subset A \cup B$ is clear.

This proves $A \cup B = B$.

"$\Leftarrow$" let $x \in A \Rightarrow x \in A \cup B \overset{\text{by assumption}}{=} B$

$\Rightarrow x \in B$

This proves $A \subset B$. $\square$

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b) Reflexivity: $x \sim x \Leftrightarrow x + x = 2x$ is even ✓

Symm.: let $x \sim y \Rightarrow x + y$ is even

$\Rightarrow y + x$ is even $\Rightarrow y \sim x$

Transitivity: let $x \sim y$ and $y \sim z$.

$\Rightarrow x + y$ even and $y + z$ even

$\Rightarrow x + y + y + z$ even $\Rightarrow 2y + x + z$ even

$\Rightarrow x + z$ even $\Rightarrow x \sim z$ $\square$

2. (a) Let the function $f : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is even} \\ x+2 & \text{if } x \text{ is odd} \end{cases}.$$

(i) (5 points) Is f injective? Prove your answer.

(ii) (5 points) Is f surjective? Prove your answer.

(b) (10 points) Use mathematical induction to prove that $n^3 \equiv n \bmod 6$ for every positive integer n.

a) i) Yes. Proof: Let $x, y \in \mathbb{Z}$ w/ $f(x) = f(y)$

Case 1) x, y even. Then $f(x) = f(y) \Leftrightarrow x = y$

" 2) x even, y odd. Then $f(x) = f(y) \Leftrightarrow \underbrace{x}_{\text{even}} = \underbrace{y+2}_{\text{odd}}$

I.e., case 2) does not occur.

Case 3) x odd, y even: same as case 2).

Case 4) x, y odd. Then $f(x) = f(y) \Leftrightarrow x+2 = y+2$

$\Leftrightarrow x = y$. □

ii) Yes. Let $y \in \mathbb{Z}$.

Case 1) y even. Then $f(y) = y$.

Case 2) y odd. Then $f(y-2) = y-2+2 = y$. □

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b) <u>$n=1$</u>: $1 = 1^3 \equiv 1 \bmod 6$ ✓

Ind. Step: $(n+1)^3 \equiv n+1 \bmod 6 \Leftrightarrow n^3+3n^2+3n+1 \equiv n+1 \bmod 6$

$\Leftrightarrow 6 \mid n^3 - n + 3n^2 + 3n$

$6 \mid n^3 - n$ by induction assumption

$\Leftrightarrow 6 \mid 3n^2+3n \Leftrightarrow 6 \mid 3(n^2+n) \Leftrightarrow 6 \mid 3n(n+1)$

Since $n(n+1)$ is always even,

$6 \mid 3n(n+1)$ <u>is</u> true. □

3. (a) (10 points) Find all integer solutions of

$$3x + 11 \equiv 9 \pmod{13}.$$

(b) (10 points) Find the solution of the following system of equations in $\mathbb{Z}_7$:

$$[1][x] + [3][y] = [4]$$
$$[3][x] + [4][y] = [2].$$

a) $3x + 11 \equiv 9 \mod 13 \Leftrightarrow 3x \equiv -2 \mod 13$

Observe: $1 = 1 \cdot 13 + (-4) \cdot 3$

$\Rightarrow -2 = (-2) \cdot 13 + 8 \cdot 3$

$\Rightarrow x = 8$ is a solution and all other solutions are of the form $8 + 13k$ with $k \in \mathbb{Z}$.

b) Multiply the first equation by $[-3]$:

$$[-3][x] + [-9][y] = [-12]$$
$$[3][x] + [4][y] = [2]$$

$+$

$\rightsquigarrow [-5][y] = [-10] = [4]$

Observe: $[-5]^{-1} = [2]^{-1} = [4]$

$\rightsquigarrow \underline{[y]} = [4] \cdot [4] = [16] = \underline{[2]}$

It remains to determine $[x]$:

$\underline{[x]} = [4] + [-3][y] = [4] + [-3][2]$

$= [-2] = \underline{[5]}$

4. (a) (10 points) Let H be the subset

$$\left\{\begin{pmatrix}1 & a\\ 0 & b\end{pmatrix} : a, b \in \mathbb{R}, b \neq 0\right\}$$

of the group of invertible 2×2 matrices with real entries. Prove that H is a subgroup of that group.

(b) (10 points) For a fixed element a of a group G, let

$$C_a = \{x \in G : ax = xa\}.$$

Prove that for all $a \in G$, the subset C_a is a subgroup of G.

a) 0) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \in H$ ✓

1) Let $\begin{pmatrix}1 & a\\ 0 & b\end{pmatrix}, \begin{pmatrix}1 & c\\ 0 & d\end{pmatrix} \in H$.

Then $\begin{pmatrix}1 & a\\ 0 & b\end{pmatrix} \cdot \begin{pmatrix}1 & c\\ 0 & d\end{pmatrix} = \begin{pmatrix}1 & c+ad\\ 0 & \underbrace{bd}_{\neq 0}\end{pmatrix} \in H$ ✓

2) Let $\begin{pmatrix}1 & a\\ 0 & b\end{pmatrix} \in H$. Then

$\begin{pmatrix}1 & a\\ 0 & b\end{pmatrix}^{-1} = \begin{pmatrix}1 & -\frac{a}{b}\\ 0 & \underbrace{\frac{1}{b}}_{\neq 0}\end{pmatrix} \in H$ ✓ □

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b) 0) Clearly, $e \in C_a$.

1) Let $x, y \in C_a$. Then $a(xy) \overset{x \in C_a}{=} xay \overset{y \in C_a}{=} (xy)a$

$\Rightarrow xy \in C_a$.

2) Let $x \in C_a$. Then $x^{-1}a = x^{-1}a\overbrace{(xx^{-1})}^{e} =$

$\overset{x \in C_a}{=} \underbrace{x^{-1}x}_{e}ax^{-1} = ax^{-1}$

$\Rightarrow x^{-1} \in C_a$. □

5. (a) (10 points) Let H_1 and H_2 be subgroups of the group G. Prove that $H_1 \cap H_2$ is a subgroup of G.

(b) (10 points) Let G be a group and $g \in G$ a fixed element. Prove that the subset

$$\langle g \rangle := \{g^n : n \in \mathbb{Z}\}$$

is a subgroup of G.

a) 0) Since $e \in H_1$ and $e \in H_2$, $e \in H_1 \cap H_2$.

1) Let $x, y \in H_1 \cap H_2$.

$\Rightarrow x, y \in H_1$ and $x, y \in H_2$.

$\Rightarrow x \cdot y \in H_1$ and $x \cdot y \in H_2 \Rightarrow xy \in H_1 \cap H_2$

2) Let $x \in H_1 \cap H_2$. $\Rightarrow x \in H_1$ and $x \in H_2$.

$\Rightarrow x^{-1} \in H_1$ and $x^{-1} \in H_2 \Rightarrow x^{-1} \in H_1 \cap H_2$.

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b) 0) $e \in \langle g \rangle$ ✓ from $g^0 = e$.

1) $g^n \cdot g^m = g^{n+m}$ ✓

2) $g^n \cdot g^{-n} = g^0 = e$ ✓