UH - Math 3330 - Dr. Heier - Spring 2014
HW 10 - Solutions to *Selected* Homework Problems
by Angelynn Alvarez

1. (Section 4.5, Problem 2) Show that

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

is a normal subgroup of the multiplicative group G of invertible matrices in $M_2(\mathbb{R})$.

Proof. First note that H contains the identity element, is closed under matrix multiplication, and contains the inverse of each of its elements. Thus, it is a subgroup of G. Now let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an invertible matrix—that is, $ad - bc \neq 0$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H$$

Similarly,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in H$$

Thus, H is a normal subgroup. □

4. (Section 4.5, Problem 22) The **center** $Z(G)$ of a group G is defined as

$$Z(G) = \{a \in G \mid ax = xa, \forall\, x \in G\}$$

Prove that $Z(G)$ is a normal subgroup of G,

Proof. We already know from previous homework problem (Section 3.3, Problem 17 in Homework 7) that $Z(G)$ is a subgroup. Thus, all we need to show is normality: Let $x \in G$ and $a \in Z(G)$. Then, $ax = xa, \forall\, x \in G$. Right-multiplication by x^{-1} yields $a = xax^{-1}$. But we already assumed $a \in Z(G)$. Thus $xax^{-1} \in Z(G)$, and thus $Z(G)$ is normal. □

5. (Section 4.5, Problem 25) Suppose H is a normal subgroup of order 2 of a group G. Prove that H is contained in $Z(G)$.

Proof. Because H is of order 2, we know that $H = \{a, e\}$ where $a \in G$ and e is the identity element of G. We already know $e \in Z(G)$, so now we need to show $a \in Z(G)$. Since H is assumed to be normal, then $gag^{-1} \in H$, for all $g \in G$. Because $H = \{a, e\}$, we have either $gag^{-1} = a$ or $gag^{-1} = e$. If $gag^{-1} = e$, we get that $ga = eg$—implying that $a = e$. This contradicts H having order 2. Thus, $gag^{-1} = a$, and $ga = ag$. Hence $a \in Z(G)$, and $H \subset Z(G)$. □

6. (Section 4.6, Problem 5) Given the Alternating group $G = A_4$, and

$$H = \{(1), (12)(34), (13)(24), (14)(23)\}$$

find the order of the quotient group G/H. Write out the distinct elements of G/H and construct a multiplication table of G/H.

Solution. The distinct elements of A_4/H are

$$A_4/H = \{H, (123)H, (234)H\}$$

Thus, $ord(G/H) = 3$. The multiplication table for G/H is

	H	**(123)H**	**(234)H**
H	H	(123)H	(234)H
(123)H	(123)H	(234)H	H
(234)H	(234)H	H	(123)H

8. (Section 4.6, Problem 13a) Let $G = S_3$. For each H that follows, show that the set of all left cosets of H *does not* for a group with respect to a product defined by $(aH)(bH) = (ab)H$.

Solution. **(a)** When $H = \{(1), (12)\}$, we have

$$G/H = \{H, (13)H, (23)H\}$$

But $((13)H)((23)H)) = ((13)(23))H = (132)H \notin G/H$. Thus, G/H is **not closed** with respect to the given product. Hence, G/H is not a group.

10. (Section 4.6, Problem 22) Let H be a normal subgroup of the group G. Prove that G/H is abelian if and only if $a^{-1}b^{-1}ab \in H$, for all $a, b \in G$.

Proof. $\boxed{\Longrightarrow}$ Assume G/H is abelian. This means that $\forall\, a, b \in G$,

$$(aH)(bH) = (bH)(aH) \Longleftrightarrow (ab)H = (ba)H$$

Left-multiplication by inverses yields: $(a^{-1}b^{-1}ab)H = H$. Hence, $a^{-1}b^{-1}ab \in H$.
$\boxed{\Longleftarrow}$ Now assume that $a^{-1}b^{-1}ab \in H$. Because $a^{-1}b^{-1} = (ba)^{-1}$, we have that $(ba)^{-1}(ab) \in H$. Thus, $(ba)^{-1}(ab)H = H$, and $(ab)H = (ba)H$. Since $(ab)H = (aH)(bH)$ and $(ba)H = (bH)(aH)$, we can conclude that $(aH)(bH) = (bH)(aH)$. Therefore, G/H is abelian, $\square$