UH - Math 3330 - Dr. Heier - Spring 2014
HW 1 - Solutions to *Selected* Homework Problems
by Angelynn Alvarez

4. (Section 1.1, Problem 34) Prove or disprove that $A \cup B = A \cup C$ implies $B = C$.

Solution. This statement is false. A counterexample is as follows: Let $A = \{0, 1\}$, $B = \{0\}$, and $C = \{1\}$. Then $A \cup B = \{0, 1\} \cup \{0\} = \{0, 1\}$, and $A \cup C = \{0, 1\} \cup \{1\} = \{0, 1\}$. Thus $A \cup B = A \cup C$, but $B \neq C$.

5. (Section 1.1, Problem 35) Prove or disprove that $A \cap B = A \cap C$ implies $B = C$.

Solution. This statement is false. A counterexample is as follows: Let $A = \{0\}$, $B = \{0, 1\}$, and let $C = \{0, 2\}$. Then $A \cap B = \{0\} \cap \{0, 1\} = \{0\}$, and $A \cap C = \{0\} \cap \{0, 2\} = \{0\}$. Thus $A \cap B = \{0\} = A \cap C$, but $B \neq C$.

6. (Section 1.2, Problem 4 c,h) Directions say to **prove all answers** for this problem.

(c) The map, $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x + 3$, is both one-to-one and onto.

Proof. Let $x, y \in \mathbb{Z}$ and assume that $f(x) = f(y)$. Then $x + 3 = y + 3$ implies that $x = y$. So f is one-to-one. Now let $b \in \mathbb{Z}$ be arbitrary. For f to be onto, we need to find an $x \in \mathbb{Z}$ such that $f(x) = b$. Let $x = b - 3$. Then $f(x) = f(b - 3) = (b - 3) - 3 = b$, as desired. Thus f is also onto. □

(h) The map, $f : \mathbb{Z} \to \mathbb{Z}, f(x) = \begin{cases} x, & \text{if } x \text{ is even} \\ x - 1, & \text{if } x \text{ is odd} \end{cases}$, is neither onto nor one-to-one.

Proof. $f(0) = 0 = f(1)$, but $0 \neq 1$. Thus f is not one-to-one. Also, if f were onto, $\mathbb{Z}$ =range(f), but $\nexists\, x \in \mathbb{Z}$ such that $f(x) = 1$. So f is not onto. □

7. (Section 1.2, Problem 7d) The map $f : A \to B$, $f(x) = |x|$, where $A = \mathbb{Z} - \{0\}$ and $B = \mathbb{Z}^+$, is onto but not one-to-one.

Proof. $f(3) = f(-3) = 3$, but $3 \neq -3$. Thus f is not one-to-one. Now let $b \in \mathbb{Z}^+$ be arbitrary. Since $f(b) = |b| = b$, f is onto. □

10. (Section 1.4, Problem 2b) Determine whether the operation is commutative or associative and whether there is an identity element. Find the inverse of each invertible element.

(b) The binary operation "$x * y = x$" is associative, but not commutative. There is no identity element.

Proof. Let $x, y, z \in \mathbb{Z}$. Then $(x * y) * z = x * z = x$ and $x * (y * z) = x * y = x$. Thus $*$ is commutative.
This operation is not commutative. For instance, let $x = 1$ and $y = 2$. Then $1 * 2 = 1$ but $2 * 1 = 2$.
Lastly, if an identity element, say e, were to exist, it would satisfy $e * x = x$ and $x * e = x$, $\forall\, x \in \mathbb{Z}$. But $e * x = e$. Thus there does not exist an identity element. □