UH - Math 3330 - Dr. Heier - Spring 2014
HW 2 - Solutions to *Selected* Homework Problems
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3. (Section 1.5, Problem 1b, 1c, 1g) For each of the following mappings $f : \mathbb{Z} \to \mathbb{Z}$, exhibit a right inverse of f with respect to the mapping composition whenever one exists.

Solution. We will use Lemma 1.25 in the textbook: A mapping f is onto $\iff$ f has a right inverse.

(b) The mapping $f(x) = 3x$ is not onto since $\nexists\ x \in \mathbb{Z}$ such that $f(x) = 1$. Thus, a right inverse does not exist.

(c) The map $f(x) = x+2$ is onto. Let $g(x) = x-2$. Then $(f \circ g)(x) = f[g(x)] = f(x-2) = (x-2)+2 = x$. Hence $g(x) = x - 2$ is the right inverse of g.

(g) The map $f(x) = \begin{cases} x, & \text{if } x \text{ is even} \\ 2x-1, & \text{if } x \text{ is odd} \end{cases}$ is not onto since $\nexists\ x \in \mathbb{Z}$ such that $f(x) = 3$. Therefore, f does not have a right inverse.

4. (Section 1.5, Problem 2b, 2c, 2g) For each of the following mappings $f : \mathbb{Z} \to \mathbb{Z}$, exhibit a left inverse of f with respect to the mapping composition whenever one exists.

Solution. We will use Lemma 1.24 in the textbook: A mapping f is one-to-one $\iff$ f has a left inverse.

(b) The mapping $f(x) = 3x$ is one-to-one because $f(x) = f(y) \iff 3x = 3y \implies x = y$. Thus, a left inverse exists. Define $g : \mathbb{Z} \to \mathbb{Z}$ by

$$g(x) = \begin{cases} \frac{x}{3}, & \text{if } x \text{ is a multiple of } 3 \\ 1, & \text{if otherwise} \end{cases}$$

Then $(g \circ f)(x) = g[f(x)] = g(3x) = \frac{(3x)}{3} = x$. Hence, g is the left inverse of f.

(c) The map $f(x) = x + 2$ is one-to-one because $f(x) = f(y) \iff x + 2 = y + 2 \iff x = y$. Thus, f has a left inverse. Define $g : \mathbb{Z} \to \mathbb{Z}$ by $g(x) = x - 2$. Then $(g \circ f)(x) = g[f(x)] = g(x + 2) = (x + 2) - 2 = x$. So g is the left inverse of f.

(g) Let us first determine if the map $f(x) = \begin{cases} x, & \text{if } x \text{ is even} \\ 2x-1, & \text{if } x \text{ is odd} \end{cases}$ is one-to-one. We consider the following cases:

Case 1: x and y are both even. $f(x) = f(y) \iff x = y$. So f is injective.
Case 2: x and y are both odd. $f(x) = f(y) \iff 2x - 1 = 2y - 1 \iff x = y$. So f is injective.
Case 3: x is even and y is odd. $f(x) = f(y) \iff x = 2y - 1$, which cannot hold since the left-hand side will always be an odd number, $\forall\ y \in \mathbb{Z}$, and thus can never equal the right-hand side (an even number). Therefore, we do not consider this case when determining the injectivity of f.

Because f is injective in the first two cases, we know that a left inverse exists. Define $g : \mathbb{Z} \to \mathbb{Z}$ by

$$g(x) = \begin{cases} x, & \text{if } x \text{ is even} \\ \frac{x+1}{2}, & \text{if } x \text{ is odd} \end{cases}$$

We now check that g is the left inverse of f. When x is even, we have $(g \circ f)(x) = g[f(x)] = g(x) = x$. When x is odd, we have $(g \circ f)(x) = g[f(x)] = g(2x-1) = \frac{(2x-1)+1}{2} = x$. In both cases, we see that $(g \circ f) = id_{\mathbb{Z}}$. So g is the left inverse of f.

5. (Section 1.5, Problem 4) Let $f : A \to A$, where A is nonempty. Prove that f has a left inverse if and only if $f^{-1}(f(S)) = S$ for every subset S of A.

Proof. $\boxed{\Longrightarrow}$ Assume that f has a left inverse, denoted by f^{-1}. This means for every subset S of A, $f^{-1}(f(S)) = (f^{-1} \circ f)(S) = id_A(S) = S$. Thus $f^{-1}(f(S)) = S$.
$\boxed{\Longleftarrow}$ Now assume that $f^{-1}(f(S)) = S$ for every subset S of A. This implies that $(f^{-1} \circ f) = id_A$. Hence f has a left inverse. □

7. (Section 1.7, Problem 1) The relations in parts **(a), (b), (d)**, and **(e)** are all relations because each element $a \in A$ gets mapped to a *unique* $b \in A$ such that (a, b) is in the relation. The relation is part **(c)** is not a mapping because the element 1 is related to three different values, i.e. $1R1$, $1R3$, and $1R5$. The relation in part **(f)** is not a mapping because 5 is related to three different values, i.e. $5R1$, $5R3$, and $5R5$.

8. (Section 1.7, Problem 2a, 2c, 2g) In each of the following parts, a relation R is defined on the set $\mathbb{Z}$. Determine in each case whether or not R is reflexive, symmetric, or transitive. **Justify your answers.**

Solution.
(a) xRy if and only if $x = 2y$
R is **not reflexive** because if $x = 1$, then $1 \neq 2(1)$.
R is **not symmetric**. Let $x = 4$ and $y = 2$. We have that $4 = 2(2)$, but $2 \neq 2(4)$.
R is **not transitive**. Let $x = 4, y = 2$ and $z = 1$. Then $4 = 2(2)$ and $2 = 2(1)$. But $4 \neq 2(1)$.

(c) xRy if and only if $y = xk$ for some $k \in \mathbb{Z}$
R **is reflexive**. We have that $x = x \cdot 1$, where $k = 1$.
R is **not symmetric**. Let $x = 1$ and $y = 3$. Then xRy because $3 = 1 \cdot 3$. But there cannot exist $k \in \mathbb{Z}$ such that $1 = 3k$.
R is **transitive**. Assume xRy and yRz. This means that $\exists\, k, k' \in \mathbb{Z}$ such that $y = xk$ and $z = yk'$. Then $z = x(kk')$, and xRz.

(g) xRy if and only if $|x| \leq |y + 1|$
R is **not reflexive.** Let $x = -2$. Then $|-2| \leq |-2+1|$ is a false statement.
R is **not symmetric.** Let $x = 2$ and $y = 4$. Then xRy because $|2| \leq |4 + 1|$. But $|4|$ is not less than or equal to $|2 + 1|$.
R is *not transitive*. Let $x = 4, y = 3$, and $z = 2$. Then xRy and yRz because $|4| \leq |3+1|$ and $|3| \leq |2+1|$. But 4 is not less than or equal to $|2 + 1|$.