UH - Math 3330 - Dr. Heier - Spring 2014
HW 6 - Solutions to *Selected* Homework Problems
by Angelynn Alvarez

3. (Section 3.1, Problem 15) Determine whether $\mathbb{Z}$ is a group with respect to $*$, and whether it is an abelian group. State which, if any conditions fail to hold.
Solution. $(\mathbb{Z}, *)$, where $x + y + 1$, is a abelian group.

Proof. We must check for closedness, associativity, existence of an identity element, and that every element has an inverse.
Let $x, y \in \mathbb{Z}$. Then by definition, $x * y = x + y + 1 \in \mathbb{Z}$. Thus, $\mathbb{Z}$ is closed under $*$.

Now let $x, y, z \in \mathbb{Z}$. Then

$$x * (y * z) = x * (y + z + 1) = x + (y + z + 1) + 1 = (x + y + 1) + z + 1 = (x * y) + z + 1 = (x * y) * z$$

So $*$ is associative.

Let $x \in \mathbb{Z}$. Then $x * (-1) = x + -1 + 1 = x + 0 = x$, and $(-1) * x = -1 + x + 1 = 0 + x = x$. Thus, the identity element of $(\mathbb{Z}, *)$ is $e = -1$.

Lastly, let $x \in \mathbb{Z}$ be arbitrary. We need to find an element y such that $x * y = -1$ and $y * x = -1$. Let $y = -x - 2$. We see that

$$x * (-x - 2) = x + (-x - 2) + 1 = -1, \quad \text{and} \quad (-x - 2) * x = (-x - 2) + x + 1 = -1$$

Thus, $y = -x - 2 = x^{-1}$, and every element in $\mathbb{Z}$ has an inverse with respect to $*$.
To check commutativity, let $x, y \in \mathbb{Z}$. Using the commutativity of addition in $\mathbb{Z}$, we have

$$x * y = x + y + 1 = y + x + 1 = y * x$$

Thus, $(Z, *)$ is an abelian group. □

4. (Section 3.1 Problem 27)
(a) Let $G = \{[a] \mid [a] \neq 0\} \subseteq \mathbb{Z}_n$. Show that G is a group with respect to multiplication in $\mathbb{Z}_n$ if and only if n is prime. State the order of G.

Proof. $\boxed{\Longrightarrow}$ For sake of contradiction, assume G is a group w.r.t. multiplication in $\mathbb{Z}_n$ but n is not prime. If n is not prime, then there exists a nontrivial factorization, $n = ab$, where $1 < a, b < n$. Considering the equivalence classes, we have $[a], [b] \in G$. Then

$$[a][b] = [ab] = [n] = [0] \notin G$$

This means G is not closed under multiplication, contradicting the fact that it is a group. ↯ Thus, n must be prime.
$\boxed{\Longleftarrow}$ This direction is very similar to proving the converse. □

The order of G is

$$\#G = \#\{\text{nonzero elements in } \mathbb{Z}_n, n \text{ prime}\} = n - 1$$

(b) Construct a multiplication table for the group G for all nonzero elements in $\mathbb{Z}_7$ and identify the inverse of each element.
Solution.

×	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[6]	[5]	[4]	[3]	[2]	[1]

We have that [1] is the inverse of itself, [6] is the inverse of itself, [2] and [4] are inverses of each other, and [3] and [5] are inverses of each other.

6. (Section 3.2, Problem 14) Let a and b be elements of a group G. Prove that G is abelian if any only if $(ab)^{-1} = a^{-1}b^{-1}$.

Proof. $\boxed{\Longrightarrow}$ Assume G is abelian. This means that $\forall\, a, b, \in G$, $ab = ba$. By Theorem 3.4(d) in the text, we know that $(ab)^{-1} = b^{-1}a^{-1}$. Since G is abelian, $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$.
$\boxed{\Longleftarrow}$ Now assume that $(ab)^{-1} = a^{-1}b^{-1}$. Using Theorem 3.4(d) again yields $(ab)^{-1} = a^{-1}b^{-1} = (ba)^{-1}$. Taking the inverse of both sides gives us

$$\left((ab)^{-1}\right)^{-1} = \left((ba)^{-1}\right)^{-1}$$

Thus $ab = ba$, and G is abelian. □

8. (Section 3.2, Problem 20) Prove or disprove that every group of order 3 is abelian.

Solution. This statement is true.

Proof. Let $G = \{a, b, e\}$ be an arbitrary group of order 3. We already know that $ea = ae = a$ and $eb = be = b$. All to show now is that $ab = e = ba$.

For sake of contradiction, assume that $ab \neq e$—say $ab = a$. Then multiplication by the inverse of a gives us $a^{-1}ab = a^{-1}a = e$. This means that $b = e$, which means G has an order less than 3. $\lightning$ Similarly, if we let $ab = b$, we get that $a = e$, which again contradicts the order of G being 3. Because G is a group, it must be closed. So if $ab \neq a$ and $ab \neq b$, then we must have $ab = e$.

Now assume that $ab = e$. Then multiplication by a^{-1} on both sides yields

$$a^{-1}ab = a^{-1}e \Longleftrightarrow (a^{-1}a)b = a^{-1}e \Longleftrightarrow b = a^{-1}$$

Multiplication by a on both sides gives us

$$ba = a^{-1}a \Longleftrightarrow ba = e$$

Thus, $ab = e = ba$, and G is abelian. □

9. (Section 3.3, Problem 2) Decide whether each of the following sets is a subgroup of $G = \{1, -1, i, -i\}$ under multiplication. If a set is not a subgroup, give a reason why it is not.

(a) The set $\{1, -1\}$ is indeed a subgroup of G.

Proof. First note that $\{1, -1\}$ is nonempty. It is also closed under multiplication. Also, the inverse of 1 is itself, and the inverse of -1 is also itself. Therefore, by Theorem 3.9, it is a subgroup. □

(b) The set $\{1, i\}$ is not a subgroup of G because $i \times i = i^2 = -1 \notin \{1, i\}$. Thus, it is not closed.

(c) The set $\{i, -i\}$ is not a subgroup of G because $i \times i = i^2 = -1 \notin \{i, -i\}$. Thus, it is not closed.

(d) The set $\{1, -i\}$ is not a subgroup of G because $(-i) \times (-i) = i^2 = -1 \notin \{1, -i\}$. Thus, it is not closed.