4/15/2014 Abstract Algebra

* HW #10 will be due in class on Thursday, April 17th
* Keep reviewing permutation group multiplication and notation

Going over today's quiz:

1) Given: $G = \{A \in M_{2\times2}(\mathbb{R}) \mid A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : ad \neq 0\}$
$H = \{A \in G \mid a = d = 1\}$

☆ Prove H is normal in G

What I did. Proven to be the wrong method. Correct method on next page.

Proof: $\forall x \in G$ s.t. $xH = Hx$ "what needs to proven for normality"

then,

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \Longrightarrow AH = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} p & q \\ 0 & s \end{bmatrix}$$

and $H = \begin{bmatrix} p & q \\ 0 & s \end{bmatrix}$

$$= \begin{bmatrix} ap & aq + bs \\ 0 & ds \end{bmatrix}$$

and

$$HA = \begin{bmatrix} p & q \\ 0 & s \end{bmatrix}\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} pa & pb + qd \\ 0 & sd \end{bmatrix}$$

now, $\det(AH) = \det(HA)$ $\Big\}$ Is this enough?

Correct form of #1:

$\forall x \in G$ s.t. $xH = Hx$

so, let $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and $H = \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix}$

then, $AH = \begin{bmatrix} a & ab'+b \\ 0 & d \end{bmatrix}$

and $HA = \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & b+b'd \\ 0 & d \end{bmatrix}$

then, $ab'+b = b+db'$ "We are told $a=d=1$"

then, $b'+b = b+b'$ ✓

$\boxed{\therefore H \text{ is normal in } G}$

Correct form of #2:

Given: H, K are both normal and $H, K \subset G$

Prove HK is also normal in G

Proof:

Let $x \in G$ and $y \in HK$

then, $y \in HK \Rightarrow \exists\, h \in H$ and $k \in K$

such that $y = hk$

Use Theorem 4.18

then, $xyx^{-1} = xhkx^{-1}$
$= xhekx^{-1}$
$= \underbrace{(xhx^{-1})}_{\in H}\underbrace{(xkx^{-1})}_{\in K}$

$\Longrightarrow (xhx^{-1})(xkx^{-1}) \in HK$

$\Longrightarrow$ By Theorem 4.18, HK is normal in G

$\boxed{\text{QED}}$

Note! : Angelynn's Method of Solving Problem #1 from today's quiz

1) $G = \{ A \in M_{2\times2}(\mathbb{R}) \,/\, A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, ad \neq 0 \}$
$H = \{ A \in G \,/\, a = d = 1 \}$. Show H is normal in G.

Proof: Let $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in G$

and $B = \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix} \in H$

then, $ABA^{-1} = \dfrac{1}{ad} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ 0 & a \end{bmatrix}$

$\Rightarrow$ then solve the above matrix product to prove H normal.

$\boxed{\text{QED}}$

Note:

$A^{-1} = \dfrac{1}{ad} \begin{bmatrix} d & -b \\ 0 & a \end{bmatrix}$

where $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

Continuitation of Ch. 4 Section 6

Recall: H is normal if $\forall x \in G$, $\underbrace{xH}_{\text{the left coset}} = Hx$

= the right coset

H is normal $\Longleftrightarrow$ $\forall x \in G$,
$h \in H$, $xhx^{-1} \in H$

H is normal: the set of cosets, G/H, form a group
with respect to operation,
$(aH)(bH) := (ab)H$

Note: "G/H" is used to indicate the quotient group

$\Rightarrow$ Every quotient group of a group G is the homomorphic image of G.

Theorem 4.23

Let $H \subset G$ be normal. Then the map will be,

$f: G \to G/H$ is an $\underbrace{\text{epimorphism}}$
$f(a) = aH$

aka "Surjective homomorphism"

by above defn. of f

Proof:

Let $a, b \in G$. Then, $f(ab) = (ab)H$

$= (aH)(bH)$

$= f(a)f(b) \Longrightarrow f$ is a homomorphism $\boxed{QED}$

same value of "a" used in "aH"

1st part of the proof.

then, $\forall aH \in G/H, \exists a \in G$ s.t.

$f(a) = aH \Longrightarrow f$ is surjective

$\Longrightarrow f$ is an epimorphism

aH is arbitrary

Remark: f is NOT injective (which means f is not a monomorphism)

Theorem 4.24 Let $f: G \to G'$ be a homomorphism. then, $Ker(f)$ is a NORMAL subgroup of G.

Proof: We will first show $Ker(f)$ is a subgroup

f is a homomorphism $\Longrightarrow f(e) = e'$, where "e" is an identity on G and "e'" is an identity on G'.

then, $e \in Ker(f) \Rightarrow Ker(f) \neq \phi$

Assume $a, b \in Ker(f)$

$\Rightarrow f(ab^{-1}) = f(a)f(b^{-1})$

$= f(a)f(b)^{-1}$

Note:

The conditions for H to be a subgroup:

1) H nonempty (aka $H \neq \phi$)

2) $a, b \in H$
$ab^{-1} \in H$

$= e' \cdot (e')^{-1}$ (because we assume $a, b \in Ker(f)$)

$= e' \cdot e' = e'$

$\Rightarrow ab^{-1} \in Ker(f) \Rightarrow Ker(f)$ is a subgroup $\boxed{QED}$

Now, to show normality:

We have only proven $Ker(f)$ is a subgroup. We still need to prove normality.

Let $x \in G$ and $a \in Ker(f)$

then, $f(xax^{-1})$

$= f(x)f(a)f(x^{-1})$ } b/c f is a homomorphism

$= f(x)f(a)f(x)^{-1}$ }

$= f(x)e'f(x)^{-1} = f(x)f(x)^{-1} = e'$

then, $xax^{-1} \in Ker(f) \Rightarrow Ker(f)$ is normal

$\boxed{QED}$

$\therefore$ $Ker(f)$ is a NORMAL SUBGROUP of G

← last step

Theorem 4.26 – Fundamental Theorem of Homomorphisms

Let $f: G \to G'$ be an <u>epimorphism</u>. Then the map will be, for $a \in G$, $\psi : G/_{Ker(f)} \to G'$

$$a \cdot Ker(f) = f(a)$$

is an isomorphism (aka bijective homomorphism)

Proof: We first show that ψ is a well-defined map ← Note: If you have 2 representations of the same element, they are mapped to the same element

We have $a, b \in G$

then, $a \cdot Ker(f) = b \cdot Ker(f)$

then, $b^{-1} a \cdot Ker(f) = Ker(f)$

then, $b^{-1} a \in Ker(f)$

$\Rightarrow f(b^{-1}a) = e'$

$\Rightarrow f(b)^{-1} \cdot f(a) = e'$

$\Rightarrow f(a) = f(b)$

$\Rightarrow \psi(a \cdot Ker(f)) = \psi(b \cdot Ker(f))$

$\Rightarrow \psi$ is then well-defined $\boxed{\text{QED}}$

4/15/2014 Vector Analysis

$$dR = \frac{dR}{dt} \cdot dt$$

Given a vector field $\vec{F}$,

curve $R = R(t) \Longrightarrow$ line integral, $\int_C \vec{F} \cdot dR = \int_C \left(\vec{F} \cdot \frac{dR}{dt}\right) dt$

surface $R = R(u,v) \Rightarrow$ surface integral, $\iint_S \left(\vec{F} \cdot \frac{\partial R}{\partial u} \times \frac{\partial R}{\partial v}\right) du\,dv$

$$dS = \left(\frac{\partial R}{\partial u} \times \frac{\partial R}{\partial v}\right) du\,dv$$