4/24/2014 Abstract Algebra

Going over Quiz #10

★ Final Exam from 11am-2pm on Thurs, May 8th in Rm 118 CBB (where our usual class is at).

1) $ab = 0$
and let $a \neq 0$

What was given: $a \neq 0$ and $ab = ac$
$\Rightarrow b = c$

then, $ab = ac$, where C is assumed to be zero (aka $c = 0$)

The proof written correctly:

$ab = 0$
let $a \neq 0$
$ab = \underbrace{0 = a \cdot 0}_{\text{based on the given hint } (0 = x \cdot 0)}$

$\Rightarrow b = 0$ ($\nexists$ zero divisors; commutative ring and unity already given)
$\therefore R$ is an integral domain

2)

Given: idempotent, $x^2 = x$ (note $x \in R$)

an element is its own square

Prove: R is an integral domain

$\Longrightarrow$ the only idempotent elements are e and 0.

so, $0^2 = 0$ and $1^2 = 1$

then, $x^2 = x \Rightarrow x^2 - x = 0$

$$\Longleftrightarrow \underbrace{x(x-1)}_{} = 0$$

now either "x" or "$x-1$" will equal 0

this is a zero divisor

then,

$$x(x-1) = 0 \Rightarrow x = 0 \text{ or }$$
$$x - 1 = 0 \Rightarrow x = 1$$

$\boxed{\therefore R \text{ is not an integral domain}}$

NOTE!: I apologize if the previous 2 proofs for Quiz #10 are incomplete or incorrectly proven.

Note: Keep watch of a mass email from Dr. Heier concerning end-of-semester info and final exam details, office hrs.

Note: Previous final exams are posted in previous Abstract Algebra sections on Dr. Heier's website

Chapter 5 Section 3 – The field of quotients of an integral domain

Motivation: $\mathbb{Z} \rightsquigarrow \mathbb{Q}$

let $\frac{a}{b}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$

a rational number

so, $\frac{a}{b} = \frac{c}{d} \iff ad = bc$

Defn 5.21 – Let D be an integral domain

Note: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$; "b" and/or "d" could be a zero divisor so to prevent zero divisors from occurring, we must then say D is an integral domain.

Continuitation of Defn 5.21 : $S = \{(a,b) \mid a, b \in D, b \neq 0\}$

then, define a ~~equivalence~~ relation, $\sim$, on S.

so, $(a,b) \sim (c,d) \iff ad = bc$

Note:

$(a,b) = a/b$

Lemma 5.22 "$\sim$" is an equivalence relation.

Proof of Lemma 5.22 : Reflexivity $\Rightarrow (a,b) \sim (a,b)$

$\iff ab = ba$ ✓

Symmetry $\Rightarrow (a,b) \sim (c,d)$

$\iff ad = bc \Rightarrow bc = ad \iff (c,d) \sim (a,b)$ ✓

Transitivity $\Rightarrow$ Let $(a,b) \sim (c,d)$ and $(c,d) \sim (f,g)$

then, $(a,b) \sim (f,g)$ } we want to find a way to get this

so, $(a,b) \sim (c,d) \Longleftrightarrow ad = bc$

and $(c,d) \sim (f,g) \Longleftrightarrow cg = df$

then, $adg = bcg$ "We are manipulating to connect $(a,b) \sim (c,d)$ and $(c,d) \sim (f,g)$".

and $bcg = bdf$

then, $adg = bdf$

$\Rightarrow ag = bf$ "Because of the Cancellation Law"

$\Longleftrightarrow (a,b) \sim (f,g)$ ✓

$\boxed{\text{QED}}$

Defn 5.23 - The sets of quotients of elements of D.

So, $Q = S/\sim$ Equivalence relation

Defn. / Lemma 5.24

Addition on Q: $[(a,b)] + [(c,d)]$

$$= \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} = \left[\left(ad+bc,\ bd\right)\right]$$

Product on $\mathbb{Q}$:

$$[(a,b)] \cdot [(c,d)] = \left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) = [(ac, bd)]$$

Theorem 5.25: Let D be an integral domain.
Then $\mathbb{Q} = S/\sim$ with "$+$" and "$\cdot$", as defined in Defn. / Lemma 5.24, is a field

Proof: Need to check all of the properties a field must have.
For example,

Note!:

$[(0,e)] = [(0,x)]$, where $x \in D \setminus \{0\}$ $\Longleftrightarrow$ $0 \cdot x = e \cdot 0$

1) 0 (zero) in $\mathbb{Q}$ is given by $[(0,b)]$ for an arbitrary $b \in D \setminus \{0\}$.

Check: $[(x,y)] + [(0,b)]$

$$= \left(\frac{x}{y}\right) + \left(\frac{0}{b}\right) = \frac{bx + y0}{by}$$

$$= \frac{bx}{by} = \frac{x}{y}$$

then, $[(x,y)] + [(0,b)] = [(bx, by)]$

$= [(x,y)]$ ✓

#5 from 4.9

Additive inverse: $-[(a,b)] = [(-a,b)]$

Check: $[(a,b)] + [(-a,b)]$

$$= \left(\frac{a}{b}\right) + \left(\frac{-a}{b}\right) = \frac{ab-ba}{b^2} = 0$$

then, $[(a,b)] + [(-a,b)] = [(ab-ba, b^2)] = 0$ ✓

Multiplicative inverse: $[(a,b)] \circ [(b,a)] = [(ab, ba)]$

$$= [(1,1)]$$

$$= \text{unity}$$

Check:

$$\left(\frac{a}{b}\right) \cdot \left(\frac{b}{a}\right) = \frac{ab}{ba} = [(ab, ba)]$$

$$= [(1,1)]$$ ✓

QED

4/24/2014 Vector Analysis

* Exam # 2 will be during class on Tuesday, April 29th (this coming Tuesday)

#5 from 4.9

$x = r\cos\theta$; $y = r\sin\theta$

Ex: $F = y^2x\hat{i} + x^2y\hat{j} + z^2\hat{k}$ In cylindrical

$\text{div } F = y^2 + x^2 + 2z = r^2 + 2z$ " "

$$\int_0^2 \int_0^{2\pi} \int_0^2 (r^2+2z)\, r \underbrace{dz\, d\theta\, dr}_{dV}$$