UH - Math 3330 - Dr. Heier - Midterm Exam - Spring 2015

Wednesday, March 25, 2015

Print your NAME: Solution

Solve all of the five problems. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit. This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. The time allowed will be announced by the proctor.

Problem 1 ________________/20 points

Problem 2 ________________/20 points

Problem 3 ________________/20 points

Problem 4 ________________/20 points

Problem 5 ________________/20 points

Total ________________/100 points

1. (a) (10 points) For sets A and B, prove the following equivalence.

$$A = B \text{ if and only if } A \cap B = A \cup B.$$

(b) (10 points) Let $x, y \in \mathbb{Z}$. Let $x \sim y$ if and only if $4 | x + 3y$. Prove that $\sim$ is an equivalence relation.

a) "$\Rightarrow$" $\underline{A \cap B} = A \cap A = A = A \cup A = \underline{A \cup B}$.

"$\Leftarrow$" Let $x \in A$. $\Rightarrow x \in A \cup B = A \cap B$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$.

We just proved: $A \subset B$. The inclusion $B \subset A$ holds by symmetry. $\Rightarrow A = B$.

---

b) • reflexive: $x \sim x \Leftrightarrow 4 | x + 3x \Leftrightarrow 4 | 4 \cdot x$ $\underline{\text{true}}$

• symm.: Let $x \sim y \Rightarrow 4 | x + 3y$

$\Leftrightarrow 4 | (x + 3y - 4 \cdot (x + y))$

$\Leftrightarrow 4 | -3x - y$

$\Leftrightarrow 4 | y + 3x \Rightarrow y \sim x$.

• transitive: Let $x \sim y$ and $y \sim z$.

$\Rightarrow 4 | x + 3y$ and $4 | y + 3z$

$\Rightarrow 4 | (x + 3y + y + 3z) \Rightarrow 4 | (x + 4y + 3z)$

$\Rightarrow 4 | x + 3z \Rightarrow \underline{x \sim z}$.

2. (a) Let the function $f : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} 2x+1 & \text{if } x \text{ is even} \\ 2x & \text{if } x \text{ is odd} \end{cases}.$$

(i) (5 points) Is f injective? Prove your answer.

(ii) (5 points) Is f surjective? Prove your answer.

(b) (10 points) Use mathematical induction to prove that $4 | 3^{2n} + 7$ for every non-negative integer n.

a) i) Observe: $\left.\begin{array}{l} f(\text{odd}) = \text{even} \\ f(\text{even}) = \text{odd} \end{array}\right\} (*)$

Let $f(x) = f(y)$. By $(*)$, either both x, y even or both x, y odd.

Case 1: x, y even. $2x+1 = 2y+1 \Rightarrow x = y$

Case 2: x, y odd. $2x = 2y \Rightarrow x = y$

Thus, f is injective.

ii) f is <u>not</u> surj., b/c, e.g., $7 \notin \text{Range}(f)$:

Assume $f(x) = 7$. $(*) \Rightarrow 2x+1 = 7 \Rightarrow x = 3 = \text{odd}$

Contradiction to $(*)$

---

b) $\underline{n=0}$: $4 | 3^0 + 7 \Leftrightarrow 4 | 1 + 7 \Leftrightarrow 4 | 8$ <u>true</u>

$P_n \rightsquigarrow P_{n+1}$: $4 | 3^{2(n+1)} + 7 \Leftrightarrow 4 | 9 \cdot 3^{2n} + 7$

$\Leftrightarrow 4 | 9 \cdot (3^{2n} + 7) - 63 + 7$

$\Leftrightarrow 4 | 9 \cdot \underbrace{(3^{2n} + 7)}_{P_n \Rightarrow \text{ divisible by } 4} - \underbrace{56}_{= 4 \cdot 14}$ <u>true</u>

3. (a) (10 points) Find all integer solutions of

$$13x \equiv 16 \pmod{25}.$$

(b) (10 points) Find the solution of the following system of equations in $\mathbb{Z}_{11}$:

$$[2][x] + [2][y] = [4]$$
$$[6][x] + [4][y] = [2].$$

a) $13x \equiv 16 \mod 25$

Observe: $1 = 2 \cdot 13 - 25$

$\Rightarrow 16 = 32 \cdot 13 - 25 \cdot 16$

Thus, $x = 32 + k \cdot 25$, $k \in \mathbb{Z}$, are all the integer solutions.

b) add $(-2) \cdot$ (Eqn 1) to (Eqn 2)

$\Rightarrow [2][x] = [-6]$

$\Rightarrow \underline{[x]} = [-3] = \underline{[8]}$

Substitute into Eqn 1:

$[2] \cdot [y] = [4] - [16] = [-12]$

$\Rightarrow \underline{[y]} = [-6] = \underline{[5]}$

4. (a) (10 points) Find the multiplicative inverse of the element [8] in $\mathbb{Z}_{51}$.

(b) (10 points) Let G be a group and H a subgroup of G. Let a be a given fixed element of G. Let

$$K = \{x \in G : \exists h \in H : x = aha^{-1}\}.$$

Prove that K is a subgroup of G.

a) $\gcd(51, 8) = \gcd(8, 51 - 6\cdot 8 = 3) =$

$= \gcd(3, 8 - 2\cdot 3 = 2) = \gcd(2, 3-2 = 1) = 1$

$\Rightarrow 1 = 3 - 2 = 3 - (8 - 2\cdot 3) = 3\cdot 3 - 8 =$

$3\cdot(51 - 6\cdot 8) - 8 = 3\cdot 51 + (-19)\cdot 8$

$\Rightarrow \underline{[8]^{-1}} = [-19] = \underline{[32]}$

---

b) 0) $e \in K$ b/c $e = a \cdot \underbrace{e}_{h} \cdot a^{-1}$. $\Rightarrow K \neq \emptyset$.

1) Let $x, y \in K$. $\Rightarrow x = ah_1a^{-1}$, $y = ah_2a^{-1}$

$\Rightarrow x \cdot y = ah_1a^{-1}ah_2a^{-1} = a\underbrace{h_1h_2}_{\in H}a^{-1} \in K$

2) Let $x \in K$, $x = aha^{-1}$.

$\Rightarrow x^{-1} = (aha^{-1})^{-1} = (a^{-1})^{-1}h^{-1}a^{-1}$

$= a\underbrace{h^{-1}}_{\in H}a^{-1} \in K$

5. (a) (10 points) Let G be a group and let H_1 and H_2 be subgroups of G. Prove that the union $H_1 \cup H_2$ is a subgroup of G if and only if $H_1 \subset H_2$ or $H_1 \supset H_2$.

(b) (10 points) Let $G = \langle a \rangle$ be a cyclic group of order n. Prove that if r divides n, then G has a subgroup of order r.

a) "$\Leftarrow$" Without loss of generality, assume $H_1 \subset H_2$ holds.

$\Rightarrow H_1 \cup H_2 = H_2 =$ subgp.

"$\Rightarrow$" Assume: <u>not</u> $(H_1 \subset H_2$ or $H_1 \supset H_2)$

$\Rightarrow H_1 \not\subset H_2$ <u>and</u> $H_2 \not\subset H_1$

$\Rightarrow \exists x \in H_1 : x \notin H_2$ and $\exists y \in H_2 : y \notin H_1$

Observe: both $x, y \in H_1 \cup H_2$, which is a subgp.

$\Rightarrow x \cdot y \in H_1 \cup H_2$.

W.l.o.g., $x \cdot y \in H_1$

$\Rightarrow y = \underbrace{x^{-1}}_{\in H_1} \cdot \underbrace{x \cdot y}_{\in H_1} \in H_1$. Contradiction.

---

b) Write $n = k \cdot r$.

Then $\langle a^k \rangle = \{e, a^k, a^{2k}, \ldots, a^{(r-1)k}\}$

is a subgp of order r.