Selected Solutions Math 4377/6308 HW1

Problem 3) Let $f: A \to B$ and $g: B \to C$ be functions. Assume that f is injective and that $g \circ f$ is injective. Does this imply that g is injective? Prove your answer.

The function g is not necessarily injective.

For example, let $A = \{1\}$, $B = \{1, 2\}$, $C = \{3\}$.

We can define $f: A \to B$ by $f(1) = 1$ and $g: B \to C$ by $g(1) = 3$, $g(2) = 3$.

Observe that f and $g \circ f$ are injective, but g is not injective.

Problem 5) Prove carefully that in any field F, all $a, b \in F$ satisfy $(-a)\cdot(-b) = a \cdot b$. Here, for any $x \in F$, $-x$ denotes the unique additive inverse of x.

Proof:

Claim 1: $0a = 0$ for all $a \in F$

① Distributive law in F

② $0 + 0 = 0$

Note, $0a + 0a \overset{①}{=} (0+0)a \overset{②}{=} 0a$

$[0a + 0a] + (-0a) = 0a + (-0a)$ [Add additive inverse of $0a$ to both sides]

$0a + [0a + -0a] = 0a + (-0a)$ [Associativity of addition in F]

$0a + 0 = 0$ [Definition 0.11 (iv) in his notes]

$0a = 0$ [Definition 0.11 (iii) in his notes]

Claim 2: $(-a)b = a(-b) = -(ab)$ for all $a, b \in F$

Observe that,

$ab + (-a)b = (a + -a)b$ [Distributive law in F]

$= 0b$ [$-a$ is the additive inverse of a]

$= 0$ [Claim 1]

Since $-(ab)$ is the unique additive inverse of ab, $-(ab) = (-a)b$

Similarly, $ab + a(-b) = a(b + -b) = a\,0 = 0$.

Therefore, $-(ab) = (-a)b = a(-b)$.

We know prove that $(-a)(-b) = ab$.

Using Claim 2 twice, $(-a)(-b) = -(a \cdot (-b)) = -(-(ab))$

If $x \in F$ then $-(-x) = x$ since $-x + x = 0$ which says x is the additive inverse of $-x$.

Therefore, $(-a)(-b) = ab$ □

Problem 7) Let $z = 1+3i$, $w = 1-i$. Write $\bar{w}$, $3z-2w$, $z\bar{w}$, $|\bar{z}|$, $\frac{w}{z}$ in the form $a+bi$.

i) $\bar{w} = \overline{1-i} = 1+i$

ii) $3z-2w = 3(1+3i)-2(1-i) = 1+11i$

iii) $z\bar{w} = (1+3i)(1+i) = 1+3i+i-3 = -2+4i$

iv) $|\bar{z}| = |\overline{1+3i}| = |1-3i| = \sqrt{1^2+(-3)^2} = \sqrt{10} + 0i$

v) $\frac{w}{z} = \frac{1-i}{1+3i} = \frac{(1-i)(1-3i)}{(1+3i)(1-3i)} = \frac{1-i-3i-3}{10} = \frac{-2}{10} + \frac{-4}{10}i = -\frac{1}{5} + \frac{-2}{5}i$

Selected Solutions for Math 4377/6308 HW2

Problem 3) Let $V = \{0\}$ consist of a single vector 0 and define $0+0=0$ and $c0 = 0$ for each scaler c in F.

Prove that V is a vector space over F.

Proof: Note, $+$ and $\cdot$ are binary operations.

i) Let $x,y \in V$. By definition of V and $+$, $x+y = 0+0 = 0+0 = y+x$.

ii) Let $x,y,z \in V$. Then,

$$\begin{aligned}(x+y)+z &= (0+0)+0 && [\text{Since } V=\{0\}]\\ &= 0+0 && [\text{Since } 0+0=0]\\ &= 0\end{aligned}$$

Similarly, $x+(y+z)=0$ which proves $(x+y)+z = x+(y+z)$ for all $x,y,z \in V$.

iii) The single vector $0 \in V$ is our zero element since for all $x \in V$, $x+0 = 0+0 = 0$.

iv) Let $x \in V$. Then, by definition of V and $+$,

$x+x = 0+0 = 0$, so x is the additive inverse of x.

v) Let $x \in V$. Since F is a field, there exists a neutral element of multiplication in F denoted by 1, then by definition of V and $\cdot$,

$$1 \cdot x = 1 \cdot 0 = 0 = x.$$

vi) Let $a,b \in F$ and $x \in V$.

Then,

$$\begin{aligned}(ab)\cdot x &= (ab)\cdot 0 && [\text{Since } V=\{0\}]\\ &= 0 && [\text{Definition of } \cdot]\end{aligned}$$

And,

$$\begin{aligned}a\cdot(b\cdot x) &= a\cdot(b\cdot 0)\\ &= a\cdot(0) && [\text{Definition of } \cdot]\\ &= 0 && [\text{Definition of } \cdot]\end{aligned}$$

Therefore, $(ab)\cdot x = a\cdot(b\cdot x)$ for all $a,b \in F$ and all $x \in V$.

vii) Let $a \in F$ and $x,y \in V$. We have,

$$\begin{aligned}a\cdot(x+y) &= a\cdot(0+0) && [\text{Since } V=\{0\}]\\ &= a\cdot 0 && [\text{Definition of } +]\end{aligned}$$

$= 0$ [Definition of $\cdot$]

In addition,

$$\begin{aligned} a\cdot x + b\cdot y &= a\cdot 0 + b\cdot 0 && [\text{Since } V=\{0\}] \\ &= 0 + 0 && [\text{Definition of } \cdot] \\ &= 0 && [\text{Definition of } +] \end{aligned}$$

Therefore, $a\cdot(x+y) = a\cdot x + a\cdot y$ for all $a\in F$ and all $x,y\in V$.

vii) Let $a,b\in F$ and $x\in V$. Then

$$\begin{aligned} (a+b)\cdot x &= (a+b)\cdot 0 && [\text{Since } V=\{0\}] \\ &= 0 && [\text{Definition of } \cdot] \end{aligned}$$

While,

$$\begin{aligned} a\cdot x + b\cdot x &= a\cdot 0 + b\cdot 0 && [\text{Since } V=\{0\}] \\ &= 0 + 0 && [\text{Definition of } \cdot] \\ &= 0 && [\text{Definition of } +] \end{aligned}$$

This shows $(a+b)\cdot x = a\cdot x + b\cdot x$ for all $a,b\in F$ and all $x\in V$.

By definition, V is a vector space over F.

Problem 6) Let $V = \{(a_1,a_2) : a_1,a_2 \in F\}$, where F is a field. Define addition of elements of V coordinatewise, and for $c\in F$ and $(a_1,a_2)\in V$, define $c(a_1,a_2) = (a_1, 0)$.

Is V a vector space over F with these operations? Justify your answer

No, axiom v from his notes is not satisfied.

Proof: By definition of V, $(1,1)\in V$ where 1 is the multiplicative neutral element of F.

By definition of scaler multiplication,

$$1(1,1) = (1,0).$$

Since F is a field, $1\neq 0$. Therefore, $(1,0)\neq(1,1)$.

This proves V is not a vector space over F with these operations.

Problem 7) Let V denote the set of ordered pair of reals. For $(a_1, a_2), (b_1, b_2) \in V$ and a real number c, define $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 2b_2)$ and $c(a_1, a_2) = (ca_1, ca_2)$. Is V a vector space with these operations?

No, vector space axiom vii from the notes does not hold.

For $2 \in \mathbb{R}$ and $(1,1), (0,1) \in V$, we have

$$2\big((1,1) + (0,1)\big) = 2(1+2(0), 1+2(1)) = 2(1,3) = (2,6)$$

and $$2(1,1) + 2(0,1) = (2,2) + (0,2) = (2, 2+4) = (2,8)$$

Since $(2,6) \neq (2,8)$, axiom vii does not hold.

This proves V is not a vector space with these operations.

Selected solutions for Math 4377/6308 HW3

Problem 3) Let W_1, W_2 be two subspaces of a vector space V. Prove that the intersection $W_1 \cap W_2$ is a subspace of V.

Proof: i) Since W_1 and W_2 are subspaces of V, $0 \in W_1$ and $0 \in W_2$. Therefore, $0 \in W_1 \cap W_2$.

ii) Let $x, y \in W_1 \cap W_2$. By definition, $x, y \in W_1$ and $x, y \in W_2$.

Since W_1 and W_2 are subspaces of V, they are closed under addition.

Therefore, $x+y \in W_1$ and $x+y \in W_2$ which implies $x+y \in W_1 \cap W_2$.

iii) Let $c \in F$ and $x \in W_1 \cap W_2$. Then, $x \in W_1$ and $x \in W_2$.

W_1 and W_2 are closed under scaler multiplication, so $cx \in W_1$ and $cx \in W_2$.

Thus, $cx \in W_1 \cap W_2$.

This proves $W_1 \cap W_2$ is a subspace of V.

Problem 4) Let W_1, W_2 be two subspaces of a vector space V. Prove that the union $W_1 \cup W_2$ is a subspace of V if and only if $W_2 \subseteq W_1$ or $W_1 \subseteq W_2$

Proof: ($\Rightarrow$) Suppose $W_1 \cup W_2$ is a subspace of V.

Let $w_1 \in W_1$ and $w_2 \in W_2$. Note, $w_1, w_2 \in W_1 \cup W_2$.

Since $W_1 \cup W_2$ is a subspace, $w_1 + w_2 \in W_1 \cup W_2$. By definition of union, $w_1 + w_2 \in W_1$ or $w_1 + w_2 \in W_2$. Therefore, $w_1 + w_2 = w$ for some $w \in W_1$ or $w_1 + w_2 = w'$ for some $w' \in W_2$.

Since W_1 and W_2 are subspaces, $w_2 = w - w_1 \in W_1$ or $w_1 = w' - w_2 \in W_2$.

This shows $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

($\Leftarrow$) Conversely, suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. By properties of sets,

$W_1 \cup W_2 = W_2$ or $W_1 \cup W_2 = W_1$.

By assumption, W_1 and W_2 are subspaces, so in either case, $W_1 \cup W_2$ is a subspace of V.

Problem 5) Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

Proof: ($\Rightarrow$) Suppose V is the direct sum of W_1 and W_2. Let $v \in V$

By definition, $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

By assumption, $v = w_1 + w_2$ for some $w_1 \in W_1$ and some $w_2 \in W_2$. Suppose there exists $w_1' \in W_1$ and $w_2' \in W_2$ such that $v = w_1' + w_2'$.

Note, $v = w_1 + w_2 = w_1' + w_2'$ and vector space operations in V give

$w_1 - w_1' = w_2' - w_2$. Since W_1 and W_2 are closed under addition,

$w_1 - w_1' \in W_1$ and $w_2' - w_2 \in W_2$. Therefore, $w_1 - w_1', w_2 - w_2' \in W_1 \cap W_2$.

By assumption, $W_1 \cap W_2 = \{0\}$. We conclude $w_1 = w_1'$ and $w_2 = w_2'$ proving

that $v = w_1 + w_2$ is the unique representation of V where $w_1 \in W_1$ and $w_2 \in W_2$.

$(\Leftarrow)$ Conversely, suppose each vector in V can be uniquely written as $v = x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$.

Note, by assumption, $V = W_1 + W_2$. Let's show $W_1 \cap W_2 = \{0\}$.

Let $x \in W_1 \cap W_2$. Since $x \in V$, by assumption, there exists unique $x_1 \in W_1$ and unique $x_2 \in W_2$

such that $x = x_1 + x_2$. We have $x \in W_1$ and $x \in W_2$ so $x = x + 0$ and $x = 0 + x$

Since x_1 and x_2 are unique, we conclude from the previous line that $x_1 = 0$ and $x_2 = 0$.

Therefore, $x = 0$ and $W_1 \cap W_2 = \{0\}$.

Selected Solutions for Math 4377/6308 HW4

Problem 3) Let S_1 and S_2 be subsets of a vector space V. Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are unequal.

Proof: Note, $\text{span}(S_1 \cap S_2) \neq \emptyset$ since if $S_1 \cap S_2 = \emptyset$ then $\text{span}(\emptyset) = \{0\}$.

Let $x \in \text{span}(S_1 \cap S_2)$. By definition of span, there exists $n \in \mathbb{N}$, $c_1, c_2, \dots, c_n \in F$, and $y_1, y_2, \dots, y_n \in S_1 \cap S_2$ such that

$$x = \sum_{i=1}^{n} c_i y_i.$$

Since $y_i \in S_1 \cap S_2$, we have $y_i \in S_1$ and $y_i \in S_2$ for all $i = 1, 2, \dots, n$.

By definition, $x \in \text{span}(S_1)$ and $x \in \text{span}(S_2)$. Therefore, $x \in \text{span}(S_1) \cap \text{span}(S_2)$.

1) Give an example where they are equal

Let $V = \mathbb{R}$ and $S_1 = S_2 = \emptyset$. Then $\text{span}(S_1 \cap S_2) = \text{span}(\emptyset) = \{0\}$ and $\text{span}(S_1) \cap \text{span}(S_2) = \text{span}(\emptyset) \cap \text{span}(\emptyset) = \{0\} \cap \{0\} = \{0\}$.

Therefore, $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$.

2) Give an example where they are not equal

Let $V = \mathbb{R}^2$, $S_1 = \left\{\binom{1}{0}, \binom{0}{1}\right\}$, and $S_2 = \left\{\binom{1}{0}, \binom{1}{1}\right\}$

Then, $\text{span}(S_1 \cap S_2) = \text{span}\left(\left\{\binom{1}{0}\right\}\right)$ and $\text{span}(S_1) \cap \text{span}(S_2) = \mathbb{R}^2 \cap \mathbb{R}^2 = \mathbb{R}^2$.

Note, $\binom{0}{1} \notin \text{span}\left(\left\{\binom{1}{0}\right\}\right)$, so $\text{span}(S_1 \cap S_2) \neq \text{span}(S_1) \cap \text{span}(S_2)$.

Problem 5) Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Proof: ($\Rightarrow$) Suppose $\{u, v\}$ is linearly dependent. By definition of linear dependence, there exists $a, b \in F$ where a and b are not both zero such that

$$au + bv = 0$$

This implies $u = -(a^{-1}b)v$ when $a \neq 0$ or $v = -(b^{-1}a)u$.

Therefore, u is a multiple of v or v is a multiple of u.

($\Leftarrow$) Conversely, suppose u or v is a multiple of the other.

By assumption, $u = av$ or $v = bu$ for some $a, b \in F$

Therefore, $u - av = 0$ or $v - bu = 0$.

Since $1 \neq 0$ in F, in either case, the set $\{u, v\}$ is linearly dependent by definition.

Problem 6) Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Proof: Suppose there exists $a, b \in \mathbb{R}$ such that

$$af + bg = \mathcal{O} \quad \text{where } \mathcal{O} \text{ is the zero function.}$$

By definition of equivalent functions in $\mathcal{F}(\mathbb{R}, \mathbb{R})$,

$$af(t) + bg(t) = \mathcal{O}(t) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Let's choose specific t values.

$t = 0: \ 0 = ae^{r0} + be^{s0} = a + b \implies a = -b$

$t = 1: \ 0 = ae^{r} + be^{s}$ (2)

Substituting and using the distributive law in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ gives,

$$a(e^r - e^s) = 0.$$

Since $r \neq s$, $e^r - e^s \neq 0$. Therefore, $a = 0$ which implies $b = 0$.

We conclude, f and g are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Selected solutions for Math 4377/6308 HW5

Problem 3) Let V be a vector space over $\mathbb{R}$. Let $v, w \in V$. Prove that if $\{v-w, v+w\}$ is linearly independent, then $\{v, w\}$ is linearly independent.

Proof: Suppose $\{v-w, v+w\}$ is linearly independent.

Let $av + bw = 0$ for some $a, b \in \mathbb{R}$.

Observe that $a = \frac{a+b}{2} + \frac{a-b}{2}$

$b = \frac{a+b}{2} - \frac{(a-b)}{2}$ since a and b are real numbers.

Substituting gives,

$$\left(\frac{a+b}{2} + \frac{a-b}{2}\right)v + \left(\frac{a+b}{2} - \frac{(a-b)}{2}\right)w = 0.$$

Using vector space properties, we find that

$$\left(\frac{a+b}{2}\right)(v+w) + \left(\frac{a-b}{2}\right)(v-w) = 0.$$

By assumption, $v+w$ and $v-w$ are linearly independent.

Therefore, $\frac{a+b}{2} = 0$ and $\frac{a-b}{2} = 0$.

This implies,

$$\frac{a+b}{2} = \frac{a-b}{2}.$$

Resulting in $2b = 0$.

Thus, $a = 0$ and $b = 0$, so $\{v, w\}$ is linearly independent.

Problem 4) For each of the following subspaces of $\mathbb{R}^5$, find a basis

(a) $W_1 = \{(a, b, c, d, e) \in \mathbb{R}^5 : a - b + c - d + e = 0\}$

Let's first write W_1 as an equivalent set.

$$W_1 = \{(b-c+d-e, b, c, d, e) \in \mathbb{R}^5 : b, c, d, e \in \mathbb{R}\}$$
$$= \{b(1,1,0,0,0) + c(-1,0,1,0,0) + d(1,0,0,1,0) + e(-1,0,0,0,1) : b,c,d,e \in \mathbb{R}\}$$

Therefore, $W_1 = \text{span}\{(1,1,0,0,0), (-1,0,1,0,0), (1,0,0,1,0), (-1,0,0,0,1)\}$.

Let's show $\{(1,1,0,0,0), (-1,0,1,0,0), (1,0,0,1,0), (-1,0,0,0,1)\}$ is a linearly independent set.

Suppose $r(1,1,0,0,0) + s(-1,0,1,0,0) + t(1,0,0,1,0) + u(-1,0,0,0,1) = (0,0,0,0,0)$ where $r, s, t, u \in \mathbb{R}$.

Then, we get the following system of equations.

$$r - s + t - u = 0$$
$$r = 0$$
$$s = 0$$
$$t = 0$$
$$u = 0.$$

Therefore, $\{(1,1,0,0,0), (-1,0,1,0,0), (1,0,0,1,0), (-1,0,0,0,1)\}$ is linearly independent and a basis by definition.

(b) $W_2 = \{(a,b,c,d,e) \in \mathbb{R}^5 : a = c \text{ and } a+b+c = d \text{ and } c+d+e = 0\}$

Let's again try to write W_2 in an equivalent form by solving:

$$a = c \quad (1)$$
$$a + b + c = d \quad (2)$$
$$c + d + e = 0 \quad (3)$$

Substituting (1) into (2) and (3) gives

$$2a + b = d \quad (2)'$$
$$a + d + e = 0 \quad (3)'$$

Substituting $(2)'$ into $(3)'$ gives,

$$e = -3a - b$$

Therefore,

$$W_2 = \{(a, b, a, 2a+b, -3a-b) : a, b \in \mathbb{R}\}$$
$$= \{a(1,0,1,2,-3) + b(0,1,0,1,-1) : a, b \in \mathbb{R}\}$$
$$= \operatorname{span}\{(1,0,1,2,-3), (0,1,0,1,-1)\}.$$

It's a quick check to verify, $\{(1,0,1,2,-3), (0,1,0,1,-1)\}$, is a linearly independent set.

By definition, $\{(1,0,1,2,-3), (0,1,0,1,-1)\}$ is a basis for W_2.

Problem 6) Let $L = \{(1,2,1,3), (0,0,1,1)\}$.

Let $G = \{v_1 = (1,2,-2,0),\ v_2 = (1,0,0,-1),\ v_3 = (0,1,1,1),\ v_4 = (1,2,2,4)\}$

You can assume without proof that G spans $\mathbb{R}^4$. Find two vectors in G that can be replaced by the two elements of L in such a way that the spanning property is preserved.

Note, L is linearly independent since if $a, b \in \mathbb{R}$ such that

$$a(1,2,1,3) + b(0,0,1,1) = (0,0,0,0)$$

then $a = 0$, $2a = 0$, $a+b = 0$, and $3a+b = 0$.

This system of equations implies $a=0$ and $b=0$.

Since G spans $\mathbb{R}^4$ by assumption and L is linearly independent, the replacement theorem guarentees we can find a subset H of G with $4-2=2$ vectors such that $H \cup L$ generates $\mathbb{R}^4$

Let's find two vectors in G that are in the span(L) to replace to guarentee the spanning property is preserved.

We have

$$\text{span}(L) = \{ r(1,2,1,3) + s(0,0,1,1) : r, s \in \mathbb{R}\}$$
$$= \{ (r, 2r, r+s, 3r+s) : r, s \in \mathbb{R}\}$$

We see that if $v \in G$ and $v \in \text{span}(L)$ then $r=1$ or $r=0$.

case 1) If $r=0$ and $v \in \text{span}(L)$ then $v = (0,0,s,s)$ where $s \in \mathbb{R}$.

Note, the v in case 1) cannot be in G.

case 2) If $r=1$ and $v \in \text{span}(L)$ then $v = (1,2,1+s,3+s)$.

By inspection of v_1, v_2, v_2, v_4, choosing $s=1$ gives v_4 and choosing $s=-3$ gives v_1.

Therefore, set $H = \{v_2, v_3\}$. Since $v_1, v_4 \in \text{span}(L)$, $\text{span}(H \cup L) = \text{span}(G)$, and we have found our two elements in G that can be replaced by the two elements in L to preserve the spanning property.

Selected Solutions to Math 4377/6308 for HW6

Problem 5) Determine explicitly the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T(1,1) = (1,1,2) \quad \text{and} \quad T(0,1) = (1,1,1).$$

Proof: Notice that $\{(1,1), (0,1)\}$ is a basis for $\mathbb{R}^2$.

Let $(x,y) \in \mathbb{R}^2$ then

$$\begin{aligned} (x,y) &= x(1,0) + y(0,1) \\ &= x\left[(1,1) - (0,1)\right] + y(0,1) \\ &= x(1,1) + (y-x)(0,1). \end{aligned}$$

Let's calculate $T(x,y)$:

$$\begin{aligned} T((x,y)) &= T\left(x(1,1) + (y-x)(0,1)\right) \\ &= x\, T((1,1)) + (y-x)\, T((0,1)) && [\text{Since } T \text{ is linear}] \\ &= x(1,1,2) + (y-x)(1,1,1) && [\text{By assumption}] \\ &= (y, y, x+y). \end{aligned}$$

Let's verify $T(1,1) = (1,1,2)$ and $T(0,1) = (1,1,1)$.

We find

$$\begin{aligned} T((1,1)) &= (1, 1, 1+1) = (1,1,2) \\ T((0,1)) &= (1, 1, 0+1) = (1,1,1). \end{aligned}$$

To show $T(x,y) = (y, y, x+y)$ is linear,

let $(a_1,a_2), (b_1,b_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}$.

Then,

$$\begin{aligned} T\left(c(a_1,a_2) + (b_1,b_2)\right) &= T\left((ca_1+b_1, ca_2+b_2)\right) \\ &= \left(ca_2+b_2, ca_2+b_2, ca_1+b_1+ca_2+b_2\right) && [\text{Definition of } T] \\ &= (ca_2, ca_2, ca_1+ca_2) + (b_2, b_2, b_1+b_2) && [\text{properties of } \mathbb{R}^2] \\ &= c(a_2, a_2, a_1+a_2) + (b_2, b_2, b_1+b_2) \\ &= c\, T((a_1,a_2)) + T(b_1,b_2). \end{aligned}$$

This proves T is a linear transformation.

Problem 7) Let V and W be vector spaces and $T: V \to W$ be linear.

(a) Prove that T is one to one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

Proof: ($\Rightarrow$) Suppose T is one to one. Let S be a linearly independent subset of V.

Note, if $S = \emptyset$, $T(S) = \emptyset$, so $T(S)$ is linearly independent.

If $S \neq \emptyset$, then $T(S) \neq \emptyset$.

Suppose $\{w_1, \ldots, w_n\} \subseteq T(S)$ such that

$$c_1 w_1 + c_2 w_2 + \cdots + c_n w_n = 0 \quad \text{where } c_1, c_2, \ldots, c_n \in F.$$

By definition of $T(S)$, for each $i = 1, \ldots, n$, there exists some $v_i \in S$ such that $T(v_i) = w_i$.

Therefore, $\sum_{i=1}^{n} c_i T(v_i) = 0$

Since T is linear,

$$T\left(\sum_{i=1}^{n} c_i v_i\right) = 0.$$

Since T is one to one,

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.$$

By assumption, S is a linearly independent set.

Therefore, $c_1 = c_2 = \cdots = c_n = 0$, and we conclude $T(S)$ is linearly independent.

($\Leftarrow$) Conversely, suppose T carries linearly independent subsets of V onto linearly independent subsets of W.

We will give two arguments:

1) Let $v \in N(T)$. Since V is a vector space [possibly infinite dimensional], V has a basis. [This is proven in Section 1.7 using Zorn's lemma].

Let β be a basis for V. Then, there exists $n \in \mathbb{N}$, $\{b_1, b_2, \ldots, b_n\} \subset \beta$, $c_1, c_2, \ldots, c_n \in F$ such that

$$v = \sum_{i=1}^{n} c_i b_i$$

Since $v \in N(T)$, $T(v) = 0$

Substituting and using linearity, $\sum_{i=1}^{n} c_i T(b_i) = 0$.

By definition of β being a basis, $\{b_1, b_2, \ldots, b_n\}$ is a linearly independent set.

By assumption, $\{T(b_1), T(b_2), \ldots, T(b_n)\}$ is linearly independent.

Therefore, $c_1 = c_2 = \cdots = c_n = 0$ which implies $v = 0$.

This shows $N(T) = \{0\}$. By Theorem 2.13 in his notes, T is one to one.

(2) Suppose by contradiction, T is not one to one.

Then, there exists $x, y \in V$ where $x \neq y$ such that $T(x) = T(y)$

Since T is linear, $T(x-y) = 0$. Since $x \neq y$, $x - y \neq 0$.

Therefore, $\{x-y\}$ is linearly independent. But, $\{0\}$ is linearly dependent

This contradicts linearly independent sets being mapped to linearly independent sets.

Therefore, T is one-to-one.

(b) Suppose that T is one to one and that S is a subset of V.

Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

Proof: ($\Rightarrow$) Suppose S is linearly independent. By part a) $T(S)$ is linearly independent.

($\Leftarrow$) Suppose $T(S)$ is linearly independent.

If $S = \emptyset$ then S is linearly independent.

If $S \neq \emptyset$ then let $\{v_1, \ldots, v_n\} \subset S$ such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \text{ where } n \in \mathbb{N} \text{ and } c_1, c_2, \ldots, c_n \in F.$$

Applying T gives,

$$T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = T(0)$$

Since T is linear,

$$c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n) = 0.$$

Since $T(S)$ is linearly independent, $c_1 = c_2 = \cdots = c_n = 0$.

Therefore, S is a linearly independent set.

(c) Suppose $\beta = \{v_1, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto.

Prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

Proof: By Theorem 2.9 in his notes, $R(T) = \operatorname{span}(T(\beta))$. Since β is linearly independent, by part a) $T(\beta)$ is linearly independent. Since T is onto, $R(T) = W$.

By definition, $T(\beta)$ is a basis for W.

Selected Solutions Math 4377/6308 HW7

Problem 4) Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is linearly independent subset of $\mathcal{L}(V, W)$

Proof: Suppose $R(T) \cap R(U) = \{0\}$.

Let $a, b \in F$ such that $aT + bU = 0$ where 0 is the zero linear transformation. By assumption, T and U are nonzero.

That is, there exists some $v_1, v_2 \in V$ such that

$$T(v_1) \neq 0$$
$$U(v_2) \neq 0.$$

Therefore,

$$aT(v_1) + bU(v_1) = 0(v_1) = 0.$$

[Note, $(aT + bU)(v_1) = aT(v_1) + bU(v_1)$ by definition of addition of functions and multiplication of a function by a scaler.]

$$\Rightarrow \quad T(a\mathbf{v_1}) = U(-b\mathbf{v_1})$$

Note, $T(av_1) \in R(T)$ and $U(-bv_1) \in R(U)$, so $T(av_1) \in R(T) \cap R(U)$.

By assumption, $T(av_1) = 0$.

Since T is linear, $aT(v_1) = 0$. From above, $T(v_1) \neq 0$, so $a = 0$.

Similarly,

$$U(-bv_2) \in R(T) \cap R(U).$$

Therefore, $-\mathbf{b}\, U(v_2) = 0$. Since $U(v_2) \neq 0$, $b = 0$.

This shows $\{T, U\}$ is a linearly independent subset in $\mathcal{L}(V, W)$.

Problem 5) Let V and W be vector spaces, and let S be a subset of V.

Define $S^\circ = \{T \in \mathcal{L}(V, W) : T(x) = 0 \text{ for all } x \in S\}$.

Prove the following statements. [What if S is the empty set? Assume below S is nonempty]

(a) S° is a subspace of $\mathcal{L}(V, W)$

proof) i) Note, for the zero transformation, $0(x) = 0$ for all $x \in S$, so $0 \in S^\circ$.

ii) Let $T_1, T_2 \in S^\circ$.

Let $x \in S$, then

$$T_1 + T_2\,(x) = T_1(x) + T_2(x) \quad \text{[Addition of two functions]}$$
$$= 0 + 0 \quad [\text{Since } T_1, T_2 \in S^\circ]$$
$$= 0$$

Therefore, $T_1 + T_2 \in S^\circ$

iii) Let $c \in F$, $T \in S^\circ$.

Then

$(cT)(x) = c\,T(x)$ [Definition of a function times a scaler]

$= c \cdot 0$ [Since $T \in S^\circ$]

$= 0$

Therefore, $cT \in S^\circ$. We conclude S° is a subspace of $\mathcal{L}(V,W)$

(b) If S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^\circ \subseteq S_1^\circ$.

proof: Suppose S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$.

If $S_1 = \emptyset$ then does $S_1^\circ = \mathcal{L}(V,W)$?

If $S_1 \neq \emptyset$ then $S_2 \neq \emptyset$.

Let $T \in S_2^\circ$ which is nonempty by part (a).

Let $x \in S_1$ then $x \in S_2$ because $S_1 \subseteq S_2$.

Since $T \in S_2^\circ$, $T(x) = 0$.

Therefore, $T(x) = 0$ for all $x \in S_1$, and $T \in S_1^\circ$ by definition.

This proves $S_2^\circ \subseteq S_1^\circ$.

(c) If V_1 and V_2 are subspaces of V, then $(V_1+V_2)^\circ = V_1^\circ \cap V_2^\circ$.

Proof: Suppose V_1 and V_2 are subspaces of V.

Claim: $(V_1+V_2)^\circ \subseteq V_1^\circ \cap V_2^\circ$

Let $T \in (V_1+V_2)^\circ$. Let $x \in V_1$.

Note, $x + 0 \in V_1 + V_2$. Since $T \in (V_1+V_2)^\circ$, $T(x+0) = 0$.

Therefore, $T(x) = T(x+0) = 0$ for all $x \in V_1$. By definition, $T \in V_1^\circ$.

Let $y \in V_2$ Using a similar proof, $T(y) = T(y+0) = 0$. Therefore, $T \in V_2^\circ$.

We conclude $T \in V_1^\circ \cap V_2^\circ$ which proves the claim.

Claim: $V_1^\circ \cap V_2^\circ \subseteq (V_1+V_2)^\circ$

Let $T \in V_1^\circ \cap V_2^\circ$ and $v \in V_1 + V_2$.

By definition of V_1+V_2, $v = x+y$ for some $x \in V_1$ and some $y \in V_2$

Since T is linear,

$$T(v) = T(x+y) = T(x) + T(y).$$

Since $T \in V_1^\circ$ and $T \in U_2^\circ$,

$$T(x) = 0 \text{ and } T(y) = 0.$$

Therefore, $T(v) = 0 + 0 = 0$.

By definition, $T \in (V_1 + U_2)^\circ$ which proves the claim.

From the two claims above, $(V_1 + U_2)^\circ = V_1^\circ \cap U_2^\circ$ completing the proof.

Problem 1) Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $T(a_1, a_2) = (a_1 + a_2, a_1 - a_2)$. Let $\beta = \{(1,0), (0,1)\}$ and $\gamma = \{(1,2), (1,1)\}$. Compute $[T]_\beta^\gamma$.

By definition of T,

$$T((1,0)) = (1, 1).$$

$$T((0,1)) = (1, -1).$$

Let's write $(1,1)$ and $(1,-1)$ as linear combinations of $(1,2)$ and $(1,1)$.

Note, $(1,1) = 0(1,2) + 1(1,1)$.

We want to find $a, b \in \mathbb{R}$ such that

$$a(1,2) + b(1,1) = (1,-1)$$

This gives the following system of linear equations:

$$a + b = 1$$

$$2a + b = -1$$

Solving gives $a = -2$ and $b = 3$.

Therefore,

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}.$$