Equilibrium States on Non-Uniformly Expanding Skew Products

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1 Introduction

Let X and Y be compact, connected Riemannian manifolds and denote by d the L_1 distance on $X \times Y$. Denote by π_X and π_Y the natural projection maps from $X \times Y$ onto X and Y, respectively. We will refer to X as the base and $\{Y_x = \{x\} \times Y\}_{x\in X}$ as the fibers of the product since

$$X \times Y = \bigcup_{x\in X} \{x\} \times Y.$$

Note that each fiber Y_x can be identified with Y. We will make the necessary distinctions as needed. Let F be a skew product on $X \times Y$; i.e. there are maps $f\colon X \to X$ and $\{g_x\colon Y_x \to Y_{fx}|\ x \in X\}$ such that

$$F(x, y) = (f(x), g_x(y)).$$

In their 1999 paper, Denker and Gordin showed that if such a fibred system is fiberwise expanding and topologically exact along fibers, then there is a unique family of fiberwise Gibbs meausures that are the conditional measures for a Gibbs measure on the product $X \times Y$. In this paper, we aim to extend this result non-uniformly expanding maps. Castro and Varandas [] proved that for non-uniformly expanding systems equipped with a certain class of Hölder continuous potentials, the Ruelle operator

$$\mathcal{L}_\varphi \psi(x, y) = \sum_{(\bar{x},\bar{y})\in F^{-1}(x,y)} e^{\varphi(\bar{x},\bar{y})}\psi(\bar{x}, \bar{y}).$$

acting on the space of α-Hölder potentials admit a unique equilibrium state μ. In Section 2, we shall describe fiberwise transfer operators $\mathcal{L}_x : C(Y_x) \to C(Y_{fx})$ defined such that for any $\psi \in C(X \times Y)$,

$$\mathcal{L}_x \psi_x(y) = \sum_{\bar{y}\in g_x^{-1} y} e^{\varphi(\bar{x},\bar{y})}\psi(\bar{x}, \bar{y}).$$

Hafouta [20] uses these operators to construct a sequential family of fiberwise measures along every fiber. In Section 3, we will use cone techniques analogous to Piraino [4] to construct a potential on

X for which $\hat{\mu} = \mu \circ \pi_X^{-1}$ is an equilibrium state, namely

$$\Phi(x) = \lim_{n \to \infty} \log \frac{\langle \mathcal{L}_x^{n+1} \mathbb{1}, \sigma \rangle}{\langle \mathcal{L}_{fx}^{n} \mathbb{1}, \sigma \rangle} \tag{1}$$

for some probability measure σ on Y. This is the same potential [4] and [5] studied in the symbolic case. With these things in hand, we will be able to prove the following.

Theorem A. *Let $(X \times Y, F)$ be a Lipschitz non-uniformly expanding skew product with fiberwise exactness. Let φ be a Hölder continuous potential on $X \times Y$ and μ be its corresponding equilibrium state. Then*

1. *the potential Φ in equation (1) exists independent of σ and $\hat{\mu} = \mu \circ \pi_X^{-1}$ is an equilibrium state for Φ, and*

2. *there is a family of fiberwise Gibbs measures $\{\mu_x \colon x \in X\}$ that form a system of conditional measures for μ relative to the partition of $X \times Y$ into vertical fibers. That is,*

$$\mu = \int_X \mu_x \, d\hat{\mu}(x).$$

Before we get into it, I'd like to take a moment to thank my advisor Vaughn Climenhaga for many insightful discussions during the writing of this paper.

2 Setting

2.1 Non-Unifomrnly Expanding Skew Products

A map $F : X \times Y \to X \times Y$ is *uniformly expanding* if there exists $C, \delta_F > 0$ and $\sigma > 1$ such that

$$d(F^n(x,y), F^n(x',y')) \geq C\gamma^n d((x,y),(x',y'))$$

whenever $d((x,y),(x',y')) \leq \delta_F$. One can assume without loss of generality that $C = 1$ by passing to an adapted metric. This reduces locally expanding to

$$d(F(x,y), F(x',y')) \geq \gamma d((x,y),(x',y'))$$

whenever $d((x,y),(x',y')) \leq \delta_F$.

We shall assume that F is a local homeomorphism and that the map $f : X \to X$ is locally uniformly expanding. Moreover, we assume there is a continuous function $(x,y) \mapsto L(x,y)$ such that, for every $(x,y) \in X \times Y$ there is a neighborhood $U_{x,y}$ of (x,y) so that $F|_{U_{x,y}}$ is invertible and

$$d(F^{-1}(u_x,u_y), F^{-1}(v_x,v_y)) \leq L(x,y)d((u_x,u_y),(v_x,v_y))$$

for all $(u_x,u_y),(v_x,v_y) \in F(U_{x,y})$. Futhermore, we assume that every point in X has the same number of preimages under f and that $|g_x^{-1}(y)|$ is constant for all $x \in X$ and $y \in Y_x$. Additionally, we shall assume that there exist constants $\gamma > 1$ and $L \geq 1$, and an open region $\mathcal{A} \subset X \times Y$ such that

1. $L(x,y) \leq L$ for every $x \in \mathcal{A}$ and $L(x,y) < \gamma^{-1}$ for all $x \notin \mathcal{A}$, and L is close to 1.
2. There exists a finite covering $\mathcal{U}$ of $X \times Y$ by open sets for which F is injective such that $\mathcal{A}$ can be covered by $q < deg(F)$. Moreover, we assume that the elements of $\mathcal{U}$ are small enough to separate curves on $X \times Y$ in the sense that if c is a distance-minimizing geodesic on $X \times Y$, then each element of $\mathcal{U}$ can intersect at most one curve in $F^{-1}(c)$.

The first condition means that F is uniformly expanding outside of $\mathcal{A}$ and not too contracting in $\mathcal{A}$. Thus, if $\mathcal{A}$ is empty, then everything is reduced to the uniformly expanding case. The second condition ensures that every point has at least one preimage in the expanding region. Note that H2' is a strengthened version of H2 from [Castro Varandas]. We need this to prove that fiberwise we are in the setting of Hafouta 20 as described below. But first we state a techincal lemma that gives us control on the distances between pre-images of points.

Lemma 2.1. *For any $n \geq 1$ and $(x, y), (x', y') \in X \times Y$, there exists a bijection between the sets of preimages $\{(\bar{x}, \bar{y}) \in X \times Y : F^n(\bar{x}, \bar{y}) = (x, y)\}$ and $\{(\bar{x}', \bar{y}') \in X \times Y : F^n(\bar{x}, \bar{y}) = (x', y')\}$. Moreover, for every $n \in \mathbb{N}$, there exists $\delta(n) > 0$ such that for every $0 < \delta \leq \delta(n)$ the distance between paired n-preimages is such that if $d((x, y), (x', y')) < \delta$, then*

$$d(F^{-n}(u_x, u_y), F^{-n}(v_x, v_y)) \leq L^n d((x, y), (x', y'))$$

for every $i = 1, \ldots, deg(F)^n$.

Proof. See Lemma 3.8 in Castro Varandas. □

Lemma 2.2. *If F satisfies H1 and $\overline{H2}, then there exists families \{L_x\}$, $\{\sigma_x\}$, and $\{q_x\}$ so that $L_x \leq L$ for some $L \geq 1$ and for each $x \in X$, we have $\sigma_x > 1$, $L_x \geq 1$, $q \in \mathbb{N}$ such that $q < deg g_x$ and for any $y, y' \in Y_{fx}$, we can pair off the preimages of $g_x^{-1}(y) = \{y_1, \ldots, y_{q_x}\}$ and $g_x^{-1}(y') = \{y'_1, \ldots, y'_{q_x}\}$ where for any $k = 1, 2, \ldots, q_x$,*

$$d_Y(y_k, y'_k) \leq L_x d_Y(y, y')$$

while for any $k = q_x + 1, \ldots, \bar{\rho}$,

$$d_Y(y_k, y'_k) \leq \sigma_x^{-1} d_Y(y, y').$$

Proof. Let $(x, y), (x', y') \in X \times Y$ and c be a distance-minimizing geodesic between the points. Let $g_x^{-1}(y) = \{x_1, \ldots, x_d\}$. Since F is a covering map, we can lift c to curves $c_1, \ldots, c_d$ such that each c_k starts at start by letting c be a distance-minimizing geodesic from x to x', then enumerate the preimages of x however you want and lift c (using the fact that T is a covering map) to curves $c_1, \ldots, c_d$ such that each c_i starts at x_i and $T(c_i) = c$. Let x_i' be the other endpoint of c_i, and then cover each c_i with "small" domains of injectivity. By (H2'), at most q of these domains can intersect A, and moreover each such domain intersects at most one of the curves c_i (this is where we need the strengthened condition, and this is what is violated in the counterexample I will show you). Thus there are at most q curves c_i that intersect A, and then applying (H1) gives the desired result.

□

2.2 Dynamics on Skew Products

To understand the dynamics of F on $X \times Y$, define for any $n \geq 0$ and $x \in X$,

$$g_x^n := g_{f^{n-1}x} \circ \cdots \circ g_x.$$

Then for any $(x, y) \in X \times Y$, the behavior of this system can be investigated through the sequence

$$F^n(x, y) = (f^n(x), g_x^n y).$$

For each $n \geq 0$, define the n^{th}-Bowen metric as

$$d_n((x, y), (x', y')) = \max_{0 \leq i \leq n} \{d(F^i(x, y), F^i(x', y'))\}.$$

Also, call $B_n((x, y), \delta) = \{(x', y') \colon d_n((x, y), (x', y')) < \delta\}$ the n^{th}-Bowen ball centered at (x, y) of radius $\delta > 0$.

We shall write $B^x(y, \varepsilon) \subset Y_x$ to denote the ball centered at $y \in Y_x$ of radius $\varepsilon > 0$. Note that

$$B^x(y, \varepsilon) = B((x, y), \varepsilon) \cap Y_x.$$

Definition 2.3. An expanding map $F\colon X\times Y\to X\times Y$ is *fiberwise exact* if for every $\varepsilon>0$ and $x\in X$, there exists $N\in\mathbb{N}$ such that $F^N B^x(y,\varepsilon)=Y_{f^N x}$ for any $y\in Y_x$.

Remark 1. An important class of examples of such maps is the case when X and Y are compact, connected manifolds. In chapter 11 of [6], it is shown that an expanding map f on a compact manifold M is topologically exact: i.e. for any open $U\subset M$, there exists $N\geq 1$ such that $f^N U=M$. Fix $x\in X$. Note that F is topologically exact. Then there exists $N\geq 0$ such that for any $y\in Y_x$, $F^N(B((x,y),\varepsilon))=X\times Y$. Fix $y\in Y$. Since $B((x,y),\varepsilon))\subset F^{-N}(X\times Y)$, there is a $y_N\in B^x(y,\varepsilon)$ such that $F^N(x,y_N)=(f^N x,y)$. Since Y is compact, N can be chosen independent of $y\in Y$. Then F is fiberwise exact.

The following lemma gives us control on the distances between pre-images of points.

Lemma 2.4. *For any $n\geq 1$ and $(x,y),(x',y')\in X\times Y$, there exists a bijection between the sets of preimages $\{(\bar{x},\bar{y})\in X\times Y\colon F^n(\bar{x},\bar{y})=(x,y)\}$ and $\{(\bar{x}',\bar{y}')\in X\times Y\colon F^n(\bar{x},\bar{y})=(x',y')\}$. Moreover, for every $n\in\mathbb{N}$, there exists $\delta(n)>0$ such that for every $0<\delta\leq\delta(n)$ the distance between paired n-preimages is such that if $d((x,y),(x',y'))<\delta$, then*

$$d(F^{-n}(u_x,u_y),F^{-n}(v_x,v_y))\leq L^n d((x,y),(x',y'))$$

for every $i=1,\dots,deg(F)^n$.

Proof. See Lemma 3.8 in Castro Varandas. □

2.3 Existence and uniqueness of equilibrium states

We say $\varphi\colon X\times Y\to\mathbb{R}$ is *α-Hölder continuous* if

$$|\varphi|_\alpha:=\sup_{(x,y),(x',y')\in X\times Y}\frac{\left|\varphi(x,y)-\varphi(x',y')\right|}{d((x,y),(x',y'))^\alpha}<\infty.$$

We denote by $C^\alpha = C^\alpha(X \times Y)$ the space of α-Hölder continuous functions on $X \times Y$. We say that φ has the *Bowen property* if φ is continuous and there is a constant K such that $\sup_{n\geq 1} |S_n\varphi(x,y) - S_n\varphi(x',y')| \leq K$.

We denote by $\mathcal{M}(X \times Y)$ the space of Borel probability measures on $X \times Y$ and $\mathcal{M}(X \times Y, F)$ those that are F-invariant. Given a continuous map $F\colon X \times Y \to X \times Y$ and a pitetnital $\varphi\colon X \times Y \to \mathbb{R}$, the variational principle asserts that

$$P_{top}(F, \varphi) = \sup\left\{h_\nu(F) + \int \varphi \, d\nu \colon \nu \in \mathcal{M}(X \times Y, F)\right\} \tag{2}$$

where $P_{top}(F, \varphi)$ denotes the topological pressure of F with respect to φ and $h_\mu(F)$ denotes the metric entropy of F. An equilibrium state for F with respect to φ is an invariant measure that achieves the supremum in the right-hand side of equation (2). Equivalently, an equilibrium state μ is an invariant probability measure that satisfies the Gibbs property: for any $\varepsilon > 0$, there exists a $C > 0$ such that

$$C^{-1} \leq \frac{\mu\big(B_n((x,y),\varepsilon)\big)}{e^{-nP(\varphi)+S_n\varphi(x,y)}} \leq C$$

for any $(x,y) \in X \times Y$ and $n \in \mathbb{N}$.

For our purposes in this paper, we fix a Hölder potential $\varphi \in C^\alpha$. We assume that φ satisfies the following

$$\sup\varphi - \inf\varphi < \varepsilon_\varphi \text{ and } |e^\varphi|_\alpha < \varepsilon_\varphi e^{\inf\varphi}$$

for some $\varepsilon_\varphi > 0$ depending only on the constants L, γ, q, and $deg(F)$.

Theorem 2.5. *Let $F\colon M \to M$ be a local homeomorphism with Lipschitz continuous inverse and $\varphi\colon M \to \mathbb{R}$ be a Hölder continuous potential satisfying (H1), (H2), and (P). Then the Ruelle-Perron-Frobenius has a spectrial gap property in the space of Hölder continuous observables, there exists a unique equilbrium state μ for F with respect to φ and the density $d\mu/d\nu$ is Hölder continuous.*

Theorem gives us a unique equilibrium state μ. Denote by $\hat{\mu} = \mu \circ \pi_X^{-1}$ the pushforward of the equilibrium state μ onto the base X. Throughout this paper, we shall refer to this measure as the transverse measure for our skew product.

2.4 Fiberwise Transfer Operators for Skew Products

As common in the literature, we will utilize Ruelle operators to study the equilibrium state on $(X \times Y, F)$. Define the transfer operator $\mathcal{L}_\varphi$ acting on $C(X \times Y)$ by sending $\psi \in C(X \times Y)$ to

$$\mathcal{L}_\varphi \psi(x, y) = \sum_{(\bar{x},\bar{y}) \in F^{-1}(x,y)} e^{\varphi(\bar{x},\bar{y})} \psi(\bar{x}, \bar{y}).$$

Note that under the skew product representation of F, we may write

$$\sum_{(\bar{x},\bar{y}) \in F^{-1}(x,y)} e^{\varphi(\bar{x},\bar{y})} \psi(\bar{x}, \bar{y}) = \sum_{\bar{x} \in f^{-1}x} \sum_{\bar{y} \in g_{\bar{x}}^{-1} y} e^{\varphi(\bar{x},\bar{y})} \psi(\bar{x}, \bar{y}).$$

This gives rise to a fiberwise transfer operator on the fibers of $X \times Y$. For every $x \in X$, let $\mathcal{L}_x : C(Y_x) \to C(Y_{fx})$ be defined by

$$\mathcal{L}_x \psi_x(y) = \sum_{\bar{y} \in g_x^{-1} y} e^{\varphi(\bar{x},\bar{y})} \psi(\bar{x}, \bar{y})$$

for any $\psi \in C(X \times Y)$. We shall iterate the transfer operator by letting

$$\mathcal{L}_x^n = \mathcal{L}_{f^{n-1}x} \circ \cdots \circ \mathcal{L}_x \colon C(Y_x) \to C(Y_{f^n x}).$$

Along with each of these fiberwise operators, we define its dual $\mathcal{L}_x^*$ by sending a probability measure $\eta \in \mathcal{M}(Y_{fx})$ to the measure $\mathcal{L}_x^* \eta \in \mathcal{M}(Y_x)$ such that for any $\psi \in C(X \times Y)$,

$$\int \psi \, d(\mathcal{L}_x^* \eta) = \int \mathcal{L}_x \psi \, d\eta.$$

3 Proof of Theorem A.1

Piraino [4] shows that for subshifts of finite type, $\hat{\mu}$ is an equilibrium state for a potential analogous to

$$\Phi(x) = \lim_{n\to\infty} \log \frac{\langle \mathcal{L}_x^{n+1}\mathbb{1}, \sigma\rangle}{\langle \mathcal{L}_{fx}^{n}\mathbb{1}, \sigma\rangle}$$

where σ is any probability measure supported on Y. We will show that this potential exists in our setting and is independent of the choice σ. Furthermore, it is Hölder continuous. We will then show that $\hat{\mu}$ is Gibbs for Φ. Thus, the push-forward $\hat{\mu}$ is the unique equilibrium state for Φ on the factor.

3.1 Birkhoff Contraction Theorem

Consider $C^\alpha = C^\alpha(X \times Y)$ as vector space over $\mathbb{R}$. A subset $\Lambda \subset C^\alpha$ is called a *cone* if $a\Lambda = \Lambda$ for all $a > 0$. A cone Λ is *convex* if $\psi + \zeta \in \Lambda$ for all $\psi,\ \zeta \in \Lambda$.We say that Λ is a closed cone if $\Lambda \cup \{0\}$ is *closed*. We assume our cones are closed, convex, and $\Lambda \cap (-\Lambda) = \emptyset$. Define a partial ordering $\preceq$ on C^α by saying $\phi \preceq \psi$ if and only if $\psi - \phi \in \Lambda \cup \{0\}$ for any $\phi, \psi \in C^\alpha$. Let

$$A = A(\phi,\psi) = \sup\{t > 0 \colon t\phi \preceq \psi\} \text{ and } B = B(\phi,\psi) = \inf\{t > 0 \colon \psi \preceq t\phi\}.$$

The *Hilbert projective metric* with respect to a closed cone Λ is defined as

$$\Theta(\phi,\psi) = \log \frac{B}{A}.$$

This definition isn't very useful for calculating distances. For that, we have the following lemma. For a proof, see Section 4 in [3].

Lemma 3.1. *Let Λ be a closed cone and Λ^* its dual. For any $\phi, \psi \in \Lambda$,*

$$\Theta(\phi,\psi) = \log\left(\sup\left\{\frac{\langle\phi,\sigma\rangle\langle\psi,\eta\rangle}{\langle\psi,\sigma\rangle\langle\phi,\eta\rangle} \colon \sigma,\eta \in \Lambda^* \text{ and } \langle\psi,\sigma\rangle\langle\phi,\eta\rangle \neq 0\right\}\right).$$

The main idea of the proof of Theorem A is to find a cone on which long occurrences of the fiberwise transfer operator is a contraction. To accomplish this, we will need the Birkhoff Contraction theorem:

Theorem 3.2. *Let Λ_1, Λ_2 be closed cones and $\mathcal{L}\colon \Lambda_1 \to \Lambda_2$ a linear map such that $\mathcal{L}\Lambda_1 \subset \Lambda_2$. Then for all $\phi, \psi \in \Lambda_1$*

$$\Theta_{\Lambda_2}(\mathcal{L}\phi, \mathcal{L}\psi) \leq \tanh\Big(\frac{diam_{\Lambda_2}(\mathcal{L}\Lambda_1)}{4}\Big)\Theta_{\Lambda_1}(\phi, \psi)$$

where $diam_{\Lambda_2}(\mathcal{L}\Lambda_1) = \sup\{\Theta_{\Lambda_1}(\phi, \psi) \colon \phi, \psi \in \mathcal{L}\Lambda_1\}$ and $\tanh\infty = 1$.

3.2 Existence and Regularity of Φ

We will use cones of the form

$$\Lambda_K = \Lambda_K^{\alpha} = \{\in C^{\alpha}(X \times Y) \colon \psi > 0 \text{ and } |\psi|_{\alpha} \leq K \inf \psi\} \cup \{0\}.$$

It can be shown that Λ_K is a closed cone in C^{α}.

Lemma 3.3. *For any $x \in X$, $\mathcal{L}_x(\Lambda_K^x) \subset \Lambda_K^{fx}$ and there exists a constant $M > 0$ such that $diam(\mathcal{L}_x\Lambda_K) \leq M < \infty$ with respect to the Hilbert projective metric.*

Proof. Fix $x \in X$. Recall the definition of s in (). Let $\delta > 0$ and $K > 0$ be so that $\beta := (1 + (1 + K)\delta)s < 1$ and $\sup_x |\phi|_{\alpha} < K\delta$. Denote by $\{y_k\}$ and $\{y'_k\}$ be the inverse images of two points y and $\bar{y}$ in Y_x, respectively. We have

$$\begin{aligned}\frac{|\mathcal{L}_x\psi(fx, y) - \mathcal{L}_x\psi(fx, y')|}{\inf \mathcal{L}_x^{\psi}} &\leq \frac{|\mathcal{L}_x\psi(fx, y) - \mathcal{L}_x\psi(fx, y')|}{de^{\inf\psi}\inf\psi} \\ &\leq d^{-1}\sum_{k=1}^{d} e^{\varphi(x,y_k)-\inf\varphi}\big|\psi(x, y_k) - \psi(x, y'_k)\big|(\inf\psi)^{-1} \\ &\qquad + d^{-1}\sum_{k=1}^{d}(\psi(y_k)/\inf\psi)e^{-\inf\varphi}\big|e^{\varphi(x,y_k)} - e^{\varphi(x,y'_k)}\big| := I_1 + I_2\end{aligned}$$

Since $d(y_k, y_k') \leq L_x d(y,y')$ for any $1 \leq k \leq q_x$ and $d(y_k, y_k') \leq \gamma_x^{-1} d(y,y')$ for all other preimages,

$$I_1 \leq d(y,y')^\alpha e^{\varepsilon_x} d^{-1}(L_x^\alpha q + (d-q)\gamma_x^{-1}) = sKd(y,y')^\alpha$$

where s is defined in ().

Next, note that $\sup \psi \leq \inf \psi + |\psi|_\alpha \leq (1+K)\inf \psi$ and

$$\left|e^{\varphi(x,y_k)} - e^{\varphi(x,y_k')}\right| \leq e^{\sup_x \varphi}|\varphi(x,y_k) - \varphi(x,y_k')| \leq e^{\inf_x \varphi + \varepsilon_x}|\varphi_x|_\alpha d(x,y_k),(x,y_k'))^\alpha.$$

Then $I_2 \leq s(1+K)\cdot \sup_x |\varphi|_\alpha$.

Thus, we have that $|\mathcal{L}_x|_\alpha \leq s(K + (1+K)\sup_x |\varphi|_\alpha)\inf \mathcal{L}_x\psi \leq sK(1+(1+K)\delta) = \beta K \inf \mathcal{L}_x\psi$. The existence of M is given by Proposition 4.3 in Castro Varandas. $\square$

Theorem 3.4. *Let* $\Phi_n(x) = \log \dfrac{\langle \mathcal{L}_x^{n+1}\mathbb{1}, \sigma\rangle}{\langle \mathcal{L}_{fx}^n \mathbb{1}, \sigma\rangle}$. *There exists* $0 < \tau < 1$ *and* $C_1 > 0$ *such that for all* $x \in X,\ n \geq 0$,

$$\left|\Phi(x) - \Phi_n(x)\right| \leq C_1\ \tau^n.$$

Proof. Let $\varepsilon > 0$ and fix $x \in X$. Let N be as in Definition 2.3. Suppose $n, m \geq k \geq N$. Then

$$\begin{aligned}\left|\Phi_n(x) - \Phi_m(x)\right| &= \left|\log \frac{\langle \mathcal{L}_x^{n+1}\mathbb{1}, \sigma\rangle}{\langle \mathcal{L}_{fx}^n \mathbb{1}, \sigma\rangle} - \log \frac{\langle \mathcal{L}_x^{m+1}\mathbb{1}, \sigma\rangle}{\langle \mathcal{L}_{fx}^m \mathbb{1}, \sigma\rangle}\right| \\ &= \left|\log \frac{\langle \mathcal{L}_{fx}^{k-1}(\mathcal{L}_x\mathbb{1}), \sigma_{fx,n}\rangle \langle \mathcal{L}_{fx}^{k-1}\mathbb{1}, \sigma_{fx,m}\rangle}{\langle \mathcal{L}_{fx}^{k-1}\mathbb{1}, \sigma_{fx,n}\rangle \langle \mathcal{L}_{fx}^{k-1}(\mathcal{L}_x\mathbb{1}), \sigma_{fx,m}\rangle}\right|\end{aligned}$$

where $\sigma_{fx,n} = (\mathcal{L}_{f^{k+1}x})^* \cdots (\mathcal{L}_{f^n x})^*\sigma$. By Lemma 3.1, we see that

$$\left|\Phi_n(x) - \Phi_m(x)\right| \leq \Theta(\mathcal{L}_{fx}^{k-1}(\mathcal{L}_x\mathbb{1}), \mathcal{L}_{fx}^{k-1}\mathbb{1}).$$

Clearly, $\mathbb{1} \in \Lambda_K$ for any $K > 0$. Then $\mathcal{L}_x\mathbb{1} \in \Lambda_{\beta K}$ by Lemma 3.3. Write $k - 1 = qN + r$ where

$0 \le r < N$ and let M be as in Lemma 3.3. Set $\eta = \tanh(M/4)$. By Lemma 3.2, we have

$$\begin{aligned}\Theta(\mathcal{L}_{fx}^{k-1}(\mathcal{L}_x\mathbb{1}), \mathcal{L}_{fx}^{k-1}\mathbb{1}) &\le \eta^{q-1}\Theta(\mathcal{L}_{fx}^{N}(\mathcal{L}_x\mathbb{1}), \mathcal{L}_{fx}^{N}\mathbb{1}) \\ &\le \eta^{q-1}M = (\eta^{1/N})^k M\eta^{-1-(r+1)/N} \\ &\le (\eta^{1/N})^k M\eta^{-2}.\end{aligned}$$

Hence, the sequence $\{\Phi_n\}_{n\ge 0}$ is Cauchy and exists at every $x \in X$. □

This proves the existence of Φ. Now we will show that Φ is Hölder continuous. Fix $\varepsilon > 0$ and $\gamma < 1$. Let $n \in \mathbb{N}$, $d_n(x, x')$ be small, and $y \in Y$. An orbit segement of length n starting from $(x, \bar{y})$ is *okay* if for all $m \in \mathbb{N}$, $F^k(x, \bar{y}) \in \mathcal{A}$ at most γm iterates. Such an orbit segement is called *good* if it is okay in hyperbolic time.

Lemma 3.5. *There is a $Q > 0$ such that for all $m \in \mathbb{N}$, if $(x, \bar{y})$ and $(x', \bar{y}')$ are in a good preimage branch, then*

$$d(F^k(x, \bar{y}), F^k(x, \bar{y})) \le Q^m e^{-2c(n-k)} d(f^n x, f^n x')$$

for all $0 \le k < n$.

Proof. Write $k = jm + i$ for $0 \le i < m$. Since our preimage branches are assumed to be good, we know

$$\begin{aligned}d(F^k(x, \bar{y}), F^k(x', \bar{y}')) &\le L^{jm} d(F^i(f^{jm}x, g_x^{jm}\bar{y}), F^i(f^{jm}x, g_{x'}^{jm}\bar{y}')) \\ &\le (L^\gamma \sigma^{-(1-\gamma)})^{jm} d(F^i(f^{jm}x, g_x\bar{y}), F^i(f^{jm}x, g_{x'}\bar{y}')) \\ &\le L^i e^{-2cjm} d((f^k x, g_x^k \bar{y}), (f^k x', g_{x'}^k \bar{y}')) \\ &\le (Le^{2c})^m e^{-2ck} d((f^k x, g_x^k \bar{y}), (f^k x', g_{x'}^k \bar{y}'))\end{aligned}$$

□

Lemma 3.6. *There are $C > 0$ and $\theta \in (0, 1)$ such that $\Sigma_b e^{S_n\varphi(x,\bar{y})} \le C\theta^m \Sigma_g e^{S_n\varphi(x,\bar{y})}$.*

Proof. If $(x, \bar{y})$ and $(x, \bar{y})$ start bad orbits of length n, then for some $jm \in \mathbb{N}$, at least γjm of the interates will be in $\mathcal{A}$. By Lemma 3.1 in Varandas Viana 2010, there are at most $Cq^{jm}e^{\varepsilon jm}d^{n-jm}$ such trajectories. Thus, in $g_x^{-n}y$, there are at most $\Sigma_j Cq^{jm}e^{\varepsilon jm}d^{n-jm}$ bad trajectories. So

$$\frac{\Sigma_b e^{S_n\varphi(x,\bar{y})}}{\Sigma_{g_x^{-n}y} e^{S_n\varphi(x,\bar{y})}} \leq e^{n(\sup\varphi - \inf\varphi)} \Sigma_j C \left(\frac{qe^{\varepsilon}}{d}\right)^{jm}$$

□

3.3 $\hat{\mu}$ is an Equilibrium State for Φ

It remains to show that the equilibrium state for Φ, μ_Φ, is equal to the transverse measure $\hat{\mu}$. The following lemma will show that the two measures are mutually absolutely continuous. Our theorem then follows since both measures are ergodic.

Fix $x \in X$ and $n \in \mathbb{N}$. Note that

$$\begin{aligned} S_n\Phi(x) &= \sum_{k=0}^{n-1} \lim_{m\to\infty} \log \frac{\langle \mathcal{L}_{x_k}^{m+1}\mathbb{1}, \sigma\rangle}{\langle \mathcal{L}_{x_{k+1}}^{m}\mathbb{1}, \sigma\rangle} \\ &= \lim_{m\to\infty} \log \frac{\langle \mathcal{L}_{x_{n-1}}^{m+1}\mathbb{1}, \sigma\rangle \cdots \langle \mathcal{L}_{x}^{m+1}\mathbb{1}, \sigma\rangle}{\langle \mathcal{L}_{x_n}^{m}\mathbb{1}, \sigma\rangle \cdots \langle \mathcal{L}_{x_1}^{m}\mathbb{1}, \sigma\rangle} = \lim_{m\to\infty} \log \frac{\langle \mathcal{L}_{x}^{m+1}\mathbb{1}, \sigma\rangle}{\langle \mathcal{L}_{x_n}^{m-n}\mathbb{1}, \sigma\rangle}. \end{aligned}$$

Thus,

$$\begin{aligned} e^{S_n\Phi(x)} &= \lim_{m\to\infty} \frac{\langle \mathcal{L}_{x}^{m+1}\mathbb{1}, \sigma\rangle}{\langle \mathcal{L}_{x_n}^{m-n}\mathbb{1}, \sigma\rangle} \\ &= \lim_{m\to\infty} \left\langle \mathcal{L}_x^n\mathbb{1}, \frac{\sigma_{x,m}}{\langle \mathbb{1}, \sigma_{x,m}\rangle} \right\rangle \\ &= \lim_{m\to\infty} \int \sum_{\overline{y}\in g_x^{-n}y} e^{S_n\varphi(x,\overline{y})}\, d\left(\frac{\sigma_{x,m}}{\langle \mathbb{1}, \sigma_{x,m}\rangle}\right) \end{aligned}$$

Choose $\varepsilon > 0$ such that $2\varepsilon < \delta_0$. Let $Q \subset Y$ be a maximal (n, ε)-separated set. Note that

$card(Q) = l < \infty$ and write $Q = \{a_1, \ldots, a_l\}$. Also, note that $Y = \cup_{a\in Q} B_n^Y(a, \varepsilon)$. Indeed, if $y \in Y$ but $y \notin \cup_{a\in Q} B_n^Y(a, \varepsilon)$, then $d_n(y, a) > \varepsilon$ for all $a \in Q$. Thus, $Q \cup \{y\}$ is (n, ε)-separated, a contradiction. To estimate the integral above, we construct an adapted partition of Y as follows. Let $Z_0 = \bigcup_{a\in Q} B_n^Y(a, \varepsilon/2)$. Define iteratively for $0 \leq k < l$ sets

$$W_{k+1} = B_n^Y(a_{k+1}, \varepsilon/2) \cup \left(B_n^Y(a_{k+1}, \varepsilon) \setminus Z_k\right)$$

and $Z_{k+1} = W_{k+1} \cup Z_k$. Note that $\{W_j\}_{j=1}^l$ is a collection of disjoint sets such that $B_n^Y(a_j, \varepsilon/2) \subset W_j \subset B_n^Y(a_j, \varepsilon)$ and $Y = \cup_{j=1}^1 B_n^Y(a_j, \varepsilon) = \sqcup_{j=1}^l W_j$. Therefore,

$$e^{S_n\Phi(x)} = \lim_{m\to\infty} \sum_{j=1}^{l} \int_{W_j} \sum_{\bar{y}\in g_x^{-n}y} e^{S_n\varphi(x,\bar{y})} \; d\left(\frac{\sigma_{x,m}}{\langle \mathbb{1}, \sigma_{x,m}\rangle}\right).$$

Lemma 3.7. *Let x, n, and ε be as above. Define*

$$\Omega_n(\varphi, \varepsilon) := \sup\Big\{\sum_{a\in S} e^{S_n\varphi(x,a)} \colon S \subset Y \;\; is \;\; (n, \varepsilon)\text{-}separated\Big\}.$$

Then $e^{S_n\Phi(x)} \geq C \; \Omega_n(\varphi, \varepsilon)$.

Proof. Fix $y \in Y$. Let N be given by fiberwise topological exactness for $\varepsilon/2$. Let $R \subset Y$ be a maximal $(n - N, \varepsilon)$-separated set. We define a map $\theta \colon R \to g_x^{-n}y$ in the following way. Let $r \in R$. By exactness, we can find a point $z' \in g_x^{-n}y$ such that $d((x_{n-N}, z'), F^{n-N}(x, r)) \leq \varepsilon/2$. Then by Lemma 2.4, there exists $z \in g_x^{-n}y$ such that $d_{n-N}((x, z), (x, r)) \leq \varepsilon/2$. The map θ is one-to-one since if $\theta(g) = \theta(g')$, then

$$d_{n-N}(g, g') \leq d_{n-N}(g, \theta(g)) + d_{n-N}(g', \theta(g')) \leq \varepsilon,$$

a contradiction since R is $(n - N, \varepsilon)$-separated.

Since φ is Hölder, it has the Bowen property: i.e. for any $\varepsilon > 0$, there exists a L if $d_n(y, y') \leq \varepsilon$, then $|S_n\varphi(x,y) - S_n\varphi(x,y')| \le L$. Thus, for each $g \in Q$, $e^{S_n\varphi(x,g)} \le e^L e^{S_n\varphi(x,\theta(g))}$. Hence,

$$\begin{aligned}\sum_{g\in Q} e^{S_{n-N}\varphi(x,g)} &\le \sum_{g\in Q} e^L e^{S_{n-N}\varphi(x,\theta(g))} \\ &\le \sum_{z\in g_x^n y} e^L e^{S_{n-N}\varphi(x,z)} \le e^{L+N\|\varphi\|} \sum_{z\in g_x^n y} e^{S_n\varphi(x,z)}\end{aligned}$$

where the first inequality holds by the Bowen property, the second by injectivity of θ, and the third because $S_n\varphi(x,y) \ge S_{n-N}\varphi(x,y) - N\|\varphi\|$ for all $(x,y) \in X \times Y$. If we let $C_1 = e^{-(L+N\|\varphi\|)}$, then

$$\sum_{z\in g_x^n y} e^{S_n\varphi(x,z)} \ge C_1 \sum_{g\in Q} e^{S_{n-N}\varphi(x,g)}. \tag{3}$$

Let $S \subset Y$ be $(n, 2\varepsilon)$-separated. For each $s \in S$, there is $r(s) \in R$ such that $d_{n-N}(s, r(s)) \le \varepsilon$. Fix $r \in R$ and let $S_r = \{s \in S |\ r = r(s)\}$. If $s \ne s' \in S_r$, then

$$d_{n-N}(s,s') \le d_{n-N}(s,r) + d_{n-N}(r,s') \le 2\varepsilon.$$

Thus, $d_N(g_x^{n-N}s, g_x^{n-N}s') > 2\varepsilon$ so $g_x^{n-N}S_r$ is $(N, 2\varepsilon)$-separated. Note that $d_n(s,s') > 2\varepsilon$ implies that $card(S_r) \le card(g_x^{n-N}S_r) < \infty$. Let M be the maximum cardinality of a $(N, 2\varepsilon)$-separated set. Hence,

$$\begin{aligned}\sum_{s\in S} e^{S_n\varphi(x,s)} &\le \sum_{s\in S} e^L e^{S_n\varphi(x,r(s))} \\ &\le \sum_{r\in R} card(S_r)\ e^{L+N\|\varphi\|}\ e^{S_{n-N}\varphi(x,r)} \\ &\le M e^{L+N\|\varphi\|} \sum_{r\in R} e^{S_{n-N}\varphi(x,r)}\end{aligned}$$

where the first inequality holds by the Bowen property and the second holds since $S_n\varphi(x,y) \le$

$S_{n-N}\varphi(x,y) + N\|\varphi\|$. Let $C_2 = M^{-1}C_1$. Then

$$\sum_{r\in R} e^{S_{n-N}\varphi(x,r)} \geq C_2 \sum_{s\in S} e^{S_n\varphi(x,s)} \tag{4}$$

Combining equations (3) and (4), we get

$$\mathcal{L}_x^n \mathbb{1} \geq C_1 \sum_{r\in R} e^{S_{n-N}\varphi(x,r)} \geq C_1 C_2 \sum_{s\in S} e^{S_n\varphi(x,s)}.$$

By Lemma 1 of [1], we know that $\Omega_n(\varphi,\varepsilon) \leq C_{\varepsilon,2\varepsilon}\ \Omega(\varphi,2\varepsilon)$. Let $C = C_{\varepsilon,2\varepsilon}^{-1} C_1 C_2$. Then

$$\begin{aligned} e^{S_n\Phi(x)} &= \lim_{m\to\infty} \sum_{j=1}^{l} \int_{W_j} \sum_{\overline{y}\in g_x^{-n}y} e^{S_n\varphi(x,\overline{y})}\ d\left(\frac{\sigma_{x,m}}{\langle \mathbb{1}, \sigma_{x,m}\rangle}\right) \\ &\geq \lim_{m\to\infty} \sum_{j=1}^{l} \int_{W_j} C\ \Omega_n(\varphi,\varepsilon)\ d\left(\frac{\sigma_{x,m}}{\langle \mathbb{1}, \sigma_{x,m}\rangle}\right) = C\ \Omega_n(\varphi,\varepsilon) \end{aligned}$$

since $\dfrac{\sigma_{x,m}}{\langle \mathbb{1}, \sigma_{x,m}\rangle}$ is a probability measure and $\{W_j\}_{j=1}^{l}$ is a collection of disjoint sets such that $Y = \sqcup_{j=1}^{l} W_j$. □

Lemma 3.8. *For any* $0 < \varepsilon < \delta_0$, *there exists* $D > 0$ *such that*

$$D^{-1} \leq \frac{\hat{\mu}\big(B_n^X(x,\varepsilon)\big)}{e^{S_n\Phi(x)}} \leq D$$

for any $x \in X$.

Proof. First, note that

$$\hat{\mu}(B_n^X(x,\varepsilon)) = \mu\circ\pi^{-1}(B_n^X(x,\varepsilon)) = \mu(B_n^X(x,\varepsilon)\times Y).$$

Let $S \subset Y$ be a maximal (n,ε)-separated set. Then S is (n,ε)-spanning in Y_x. Let $(x',y') \in \overline{B_n^X(x,\varepsilon)\times Y}$. Choose $s\in S$ such that $d_n^Y(s,y') < \varepsilon$. The triangle inequality shows that $d_n((x,s),(x',y')) \leq$

2ε. Thus, the set $\{(x,s)\colon s\in S\}$ is $(n,2\varepsilon)$-spanning in $\overline{B_n^X(x,\varepsilon)\times Y}$. So

$$\hat{\mu}(B_n^X(x,\varepsilon)) \le \sum_{s\in S}\mu(B_n((x,y),2\varepsilon)) \le C_{2\varepsilon}\sum_{s\in S} e^{S_n\varphi(x,s)} \le De^{S_n\Phi(x)} \tag{5}$$

where the second inequality is holds by the Gibbs property of μ at scale 2ε and the third holds by lemma 3.7.

Choose a (n,ε)-separated set $R\subset Y$ and consider the collection $\{B_n((x,r),\varepsilon/2)\colon r\in R\}$. Then

$$\hat{\mu}\Big(B_n^X(x,\varepsilon)\Big) \ge \mu\Big(B_n\big((x,s),\varepsilon/2\big)\Big) \ge C_{\varepsilon/2}^{-1}\sum_{s\in S} e^{S_n\varphi(x,s)}.$$

Since this holds for an arbitrary (n,ε)-separated set, $\hat{\mu}\big(B_n^X(x,\varepsilon)\big) \ge C_{\varepsilon/2}^{-1}\Omega_n(\varphi,\varepsilon)$. Note that $g_x^{-n}y$ is (n,ε)-separated for all $y\in Y$. Then

$$\begin{aligned} e^{S_n\Phi(x)} &= \lim_{m\to\infty}\sum_{j=1}^{l}\int_{W_j}\sum_{\bar{y}\in g_x^{-n}y} e^{S_n\varphi(x,\bar{y})}\; d\bigg(\frac{\sigma_{x,m}}{\langle \mathbb{1},\sigma_{x,m}\rangle}\bigg) \\ &\le \Omega_n(x,\varepsilon) \le C_{\varepsilon/2}^{-1}\;\hat{\mu}\Big(B_n^X(x,\varepsilon)\Big). \end{aligned} \tag{6}$$

Combining equations (5) and (6) shows that $\hat{\mu}$ is a Gibbs measure for Φ. □

Lemma 3.8 shows that $\hat{\mu}$ is a Gibbs measure for Φ. Note that this implies that $P(\Phi)=0$.[1]

4 Fiber Measures are Conditionals

Theorem A allows us to use properties of $\hat{\mu}$ as an equilibrium state via a transfer operator on X. To this end, let $\mathcal{L}_\Phi\colon C(X)\to C(X)$ be defined by

$$\mathcal{L}_\Phi\xi(x) = \sum_{\bar{x}\in f^{-1}x}\lambda_{\bar{x}}\;\xi(\bar{x})$$

for any $\xi\in C(X)$. Lemma **??** allows use to use the following theorem. See [7] for details.

[1] Should this be $P(\Phi)=P(\varphi)$?

Theorem 4.1. *For any Hölder* $\Phi\colon X \to \mathbb{R}$*, the following hold:*

1. *There exists a real number* $\hat{\lambda} > 0$ *and* $\hat{\nu} \in \mathcal{M}(X)$ *such that* $\mathcal{L}_\Phi^* \hat{\nu} = \hat{\lambda}\hat{\nu}$*.*

2. *There exists a unique* $\hat{h} \in C(X)$ *such that* $\mathcal{L}_\Phi \hat{h} = \hat{\lambda}\hat{h}$*.*

To completely understand the equilibrium state μ on $(X \times Y, F)$, we need to understand how it gives weight to the fibers $\{Y_x\}_{x\in X}$. Mayer [2] uses the fiberwise transfer operators to construct families of measures that satisfy the following theorem for almost everywhere $x \in X$. However, we will need it to hold along every fiber.

Theorem 4.2. *For any Hölder* $\varphi\colon X \times Y \to \mathbb{R}$ *and its associated family of random transfer operators* $\{\mathcal{L}_x\}_{x\in X}$*, the following hold:*

1. *There exists a unique family of probability measures* $\nu_x \in \mathcal{M}(Y_x)$ *such that for all* $x \in X$*,* $\mathcal{L}_x^* \nu_{fx} = \lambda_x \nu_x$ *where* $\lambda^x = \nu_{fx}(\mathcal{L}_x \mathbb{1})$*.*

2. *There exists a unique* α*-Hölder continuous function* $h\colon X \times Y \to X \times Y$ *such that for all* $x \in X$*,*

$$\mathcal{L}_x h_x = \lambda_x h_{fx} \quad \text{and} \quad \nu_x(h_x) = 1.$$

3. *Let* $\bar{\varphi}_x = \varphi_x + \log h_x - \log h_{fx} \circ g_x - \log \lambda_x$ *and denote by* $\bar{\mathcal{L}}_x$ *the normalized transfer operator on* Y_x*. Let* $\mu_x = h_x \nu_x$*. For every* $x \in X$*,*

   (a) $\bar{\mathcal{L}_x}^* \mu_{fx} = \mu_x$*, and*

   (b) *for all* $\psi \in C(X \times Y)$*,* $\bar{\mathcal{L}}_x^n \psi \to \int \psi \, d\mu_x$ *exponentially as* $n \to \infty$*.*

In light of Lemma [H2' implies Hafouta], the proof of this follows from [Hafouta].

We wish to use $\hat{\mu}$ and its corresponding family of measures $\{\mu_x\}_{x\in X}$ to build a measure on $X \times Y$. To do this, we first prove the following lemmas.

Lemma 4.3. *For any* $\psi \in C(X \times Y)$*, the map* $x \mapsto \mathcal{L}_{\varphi}^{x} \psi_x$ *is continuous with respect to the Usual topology.*

Proof. Suppose $\psi \in C(X \times Y)$. Let $0 < d_X(x,x') < \delta_0$ and $y \in Y$ be fixed. Then

$$\begin{aligned}\left|\bar{\mathcal{L}_x}\psi_x(y) - \bar{\mathcal{L}_{x'}}\psi_{x'}(y)\right| &\le \sum_{z \in g_x^{-1}y} \left(e^{\overline{\varphi}(x,z)}\left|\psi(x,z) - \psi(x',z')\right| + \|\psi\|_\infty \left|e^{\overline{\varphi}(x,z)} - e^{\overline{\varphi}(x',z')}\right|\right) \\ &\le M_1 \sum_{z \in g_x^{-1}y} e^{\overline{\varphi}(x,z)} + \|\psi\|_\infty \sum_{z \in g_x^{-1}y} \left|e^{\overline{\varphi}(x,z)} - e^{\overline{\varphi}(x',z')}\right|\end{aligned}$$

where $M_1 = \sup\{\left|\psi(x,z) - \psi(x',z')\right| : d\big((x,z),(x',z')\big) < \delta_0\}$. Part (2) of Theorem 4.2 implies that $\bar{\mathcal{L}_x}\mathbb{1} = \mathbb{1}$. This along with the argument in the paragraph above shows that

$$\left|\bar{\mathcal{L}_x}\psi_x(y) - \bar{\mathcal{L}_{x'}}\psi_{x'}(y)\right| \le M_1 + \|\psi\|_\infty \left(e^{C_{\overline{\varphi}} d_X(x,x')^\alpha} - 1\right) \to 0 \quad \text{as } d_X(x,x') \to 0.$$

This finishes the proof. □

Lemma 4.4. *For every continuous $\psi \colon X \times Y \to \mathbb{R}$, the map $x \mapsto \nu_x(\psi_x)$ is measurable.*

Proof. Fix $x \in X$ and let $y \in Y$. Define

$$\nu_{x,n} = \frac{(\mathcal{L}_x^n)^* \delta_{(f^n x, y)}}{\mathcal{L}_x^n \mathbb{1}(f^n x, y)}$$

where δ is the Dirac measure at a point in the product. Then by item $3(b)$ of Theorem 4.2, for any $\psi \in C(X \times Y)$, we have

$$\lim_{n\to\infty} \nu_{x,n}(\psi_x) = \lim_{n\to\infty} \frac{\mathcal{L}_x^n \psi(f^n x, y)}{\mathcal{L}_x^n \mathbb{1}(f^n x, y)} = \lim_{n\to\infty} \frac{\mathcal{L}_x^n(\psi_x/h_x)(f^n x, y)}{\mathcal{L}_x^n(\mathbb{1}/h_x)(f^n x, y)} = \frac{\nu_x(\psi_x)}{\nu_x(\mathbb{1})} = \nu_x(\psi_x).$$

Thus, $\nu_{x,n} \xrightarrow{n\to\infty} \nu_x$ in the weak* topology. The measurability of $x \mapsto \nu_x(\psi_x)$ then follows from Lemma 4.3. □

Thus, we can define a measure on $X \times Y$ by $d\nu(x,y) = d\nu_x(y)d\hat{\nu}(x)$. Theorems 4.1 and 4.2 allow us the following results about this measure ν.

Lemma 4.5. *Let $\hat{\eta} \in \mathcal{M}(X,f)$ and $\{\eta_x\}_{x\in X}$ be given by Theorem 4.2. If $\mathcal{L}_\Phi^* \hat{\eta} = \lambda\hat{\eta}$ for some $\lambda > 0$, then $d\eta = d\eta_x d\hat{\eta}$ satisfies $\mathcal{L}_\varphi^* \eta = \lambda\eta$.*

Proof.

$$\begin{aligned}
\int_{X\times Y} \mathcal{L}_\varphi \psi(x,y)\ d\eta(x,y) &= \int_X \int_{Y_x} \sum_{\bar{x}\in f^{-1}x} \sum_{\bar{y}\in g_{\bar{x}}^{-1} y} e^{\varphi(\bar{x},\bar{y})}\psi(\bar{x},\bar{y})\ \ d\eta_x(y) d\hat{\eta}(x) \\
&= \int_X \left(\sum_{\bar{x}\in f^{-1}x} \int_{Y_{\bar{x}}} \mathcal{L}^{\bar{x}} \psi(\bar{x},y)\ \ d\eta_x(y) \right) d\hat{\eta}(x) \\
&= \int_X \left(\sum_{\bar{x}\in f^{-1}x} \lambda_{\bar{x}} \int_{Y_{\bar{x}}} \psi(\bar{x},y)\ \ d\eta_{\bar{x}}(y) \right) d\hat{\eta}(x) \\
&= \int_X \mathcal{L}_\Phi \left(\int_{Y_x} \psi(x,y)\ \ d\eta_x(y) \right) d\hat{\eta}(x) \\
&= \lambda \int_X \int_{Y_x} \psi(x,y)\ \ d\eta_x(y) d\hat{\eta}(x)
\end{aligned}$$

□

Note that this implies that $P(\varphi) = P(\Phi)$.

Lemma 4.6. *Let $\hat{h}$ and $\hat{\nu}$ be as in 4.1 and consider the measure $d\hat{\mu} = \hat{h} d\hat{\nu}$ on X. Let $\{h_x\}$ be given by item 2 of Theorem 4.2. Then the function $h(x,y) = \hat{h}(x)\bar{h}_X(y)$ satisfies $\mathcal{L}_\varphi h = \lambda h$.*

Proof.

$$\begin{aligned}
\sum_{(\bar{x},\bar{y})\in F^{-1}(x,y)} e^{\varphi(\bar{x},\bar{y})}\ h(x,y) &= \sum_{\bar{x}\in f^{-1}(x)} \hat{h}(\bar{x})\ \mathcal{L}_{\bar{x}} \bar{h}_{\bar{x}}(y) \\
&= \sum_{\bar{x}\in f^{-1}(x)} \lambda_{\bar{x}} \hat{h}(\bar{x})\ \bar{h}_x(y) = \bar{h}_x(y)\ \mathcal{L}_\Phi \hat{h}(x) = \lambda h(x,y)
\end{aligned}$$

□

Lemmas 4.5 and 4.6 give eigendata for $\mathcal{L}_\varphi$. It is well known that this data is uniquely determined by the RPF Theorem and that the unique equilibrium state is the measure

$$d\mu(x,y) = \hat{h}(x)\bar{h}_x(y)\ d\nu_x(y) d\hat{\nu}(x) = d\mu_x(y) d\hat{\mu}(x).$$

This finishes the proof of Theorem B.

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