Calc 1 Review

Let $f(x) = \begin{cases} 1-x & \text{if } -1 \le x < 0 \\ 2x^2-2 & 0 \le x \le 1 \\ -x+2 & 1 < x < 2 \\ 1 & x = 2 \\ 2x-4 & 2 < x \le 3 \end{cases}$.

1) $\lim_{x\to 2} f(x) = 0$ since

$\lim_{x\to 2^-} f(x) = -2+2 = 0$

$\lim_{x\to 2^+} f(x) = 2(2)-4 = 0$

2) $\lim_{x\to 1^+} f(x) = -1+2 = 1$

3) $\lim_{x\to 0^-} f(x) = 1-0 = 1$

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4) Evaluate $\lim_{x\to 2} \frac{x-2}{x^2-4}$.

$\lim_{x\to 2} \frac{x-2}{x^2-4} = \frac{0}{0}$ indeterminant

$\lim_{x\to 2} \frac{\cancel{x-2}}{(x+2)\cancel{(x-2)}} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

5) Evaluate $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$.

$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \frac{0}{0}$ indeterminant

$\frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}}$

$\Rightarrow \lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4}(\sqrt{x}+2) = 4$

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6) Find $\lim_{x\to 6^+} f(x)$ if $f(x) = \begin{cases} x^2+x+1 & \text{if } x < -1 \\ x^2-x & -1 \le x \le 6 \\ x-7 & x > 6 \end{cases}$

$\lim_{x\to 6^+} f(x) = 6-7 = -1$

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7) Given $\lim_{x\to 2} f(x) = 5$ and $\lim_{x\to 2} g(x) = -1$. Find $\lim_{x\to 2}[2f(x) - g(x)]$.

$\lim_{x\to 2} [2f(x) - g(x)] = 2\lim_{x\to 2} f(x) - \lim_{x\to 2} g(x) = 2(5)-(-1) = 10+1 = 11$

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8) Determine if $f(x)$ is continuous at $x=-1$ and at $x=6$ if

$f(x) = \begin{cases} 1-x & \text{if } x < -1 \\ x^2-x & -1 \le x \le 6 \\ x-7 & x > 6 \end{cases}$.

$f(-1) = (-1)^2-(-1) = 1+1 = 2$

$f(6) = 6^2-6 = 30$

$\Rightarrow f(-1)$ and $f(6)$ are defined

$\lim_{x\to -1^+} f(x) = (-1)^2-(-1) = 2$

$\lim_{x\to -1^-} f(x) = 1-(-1) = 2$

$\Rightarrow \lim_{x\to -1} f(x)$ exists and equals 2

Therefore, f is continuous at $x=-1$ since $\lim_{x\to -1} f(x) = 2 = f(-1)$

$\lim_{x\to 6^-} f(x) = 6^2-6 = 30$

$\lim_{x\to 6^+} f(x) = 6-7 = -1$

So $\lim_{x\to 6} f(x)$ DNE $\Rightarrow f$ is not continuous at $x=6$

9) $\lim_{x\to 0} \frac{\sin(5x)}{2x} = \frac{5}{2}\lim_{x\to 0} \frac{\sin(5x)}{5x} = \frac{5}{2}$

10) $\lim_{x\to 0} \frac{\tan^2(5x)}{2x^2} = \frac{1}{2}\lim_{x\to 0} \frac{\tan^2(5x)}{x^2}$

$= \frac{1}{2}\left(\lim_{x\to 0} \frac{\tan 5x}{x}\right)^2 = \frac{1}{2}\left(5\lim_{x\to 0} \frac{\sin 5x}{5x}\cdot\frac{1}{\cos 5x}\right)^2$

$= \frac{25}{2}\left(\lim_{x\to 0} \frac{\sin 5x}{5x}\right)^2\left(\lim_{x\to 0} \frac{1}{\cos x}\right)^2 = \frac{25}{2}$

11) $\lim_{x\to 0} \frac{x}{\tan(2x)} = \lim_{x\to 0} \frac{x}{1}\cdot\frac{\cos(2x)}{\sin(2x)}$

$= \left(\lim_{x\to 0} \frac{x}{\sin(2x)}\right)\left(\lim_{x\to 0} \cos(2x)\right)$

$= \frac{1}{2}\lim_{x\to 0} \frac{2x}{\sin(2x)} = \frac{1}{2}$

12) Determine the value of A that makes $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ Ax+3 & -1 \le x \end{cases}$ continuous at $x = -1$.

$\lim_{x\to -1^-} f(x) = (-1)^2 = A(-1)+3 = f(-1) \Rightarrow 1 = -A+3$ or $\boxed{A=2}$

13) Determine values of B and C that makes $f(x) = \begin{cases} Bx-C & \text{if } x \le 1 \\ 4x & 1 < x < 2 \\ Cx^2-B & x \ge 2 \end{cases}$ continuous everywhere.

$\lim_{x\to 1^+} f(x) = 4(1) = B(1)-C$ $\quad \lim_{x\to 2^-} f(x) = 4(2) = C(2)^2-B$

Therefore, our choice of B and C should satisfy $B-C=4$, $-B+4C=8$

Solving by elimination, $\cancel{B}-C=4$, $-\cancel{B}+4C=8$

$3C = 12 \Rightarrow \boxed{C=4}$

So $B-4=4 \Rightarrow \boxed{B=8}$

14) Does the intermediate value theorem guarantee a solution to $2x^3+x^2+3x-1=0$?

$2(0)^3+(0)^2+3(0)-1=-1$

$2(2)^3+(2)^2+3(2)-1=25$

Polynomials are continuous on $\mathbb{R}$ and $-1<0<25$ so IVT guarentees a solution on $[0,2]$.

15) Does the intermediate value theorem guarantee a solution to $\frac{x^2+4}{x-2}=0$ on the interval $[0,4]$?

$\frac{0^2+4}{0-2} = -2 \qquad \frac{4^2+4}{4-2} = \frac{20}{2} = 10$

$-2<0<10$

However, there is a V.A. at $x=2$ so function is not cont. on $[0,4]$.
IVT does not apply.

Use the definition of derivative to find..

16) $f'(2)$ when $f(x)=\frac{1}{x-1}$

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{\frac{1}{h+1}-1}{h}\cdot\frac{h+1}{h+1}$$

$$=\lim_{h\to 0}\frac{1-(h+1)}{h(h+1)}=\lim_{h\to 0}\frac{-1}{h+1}=-1$$

17) $f'(x)$ when $f(x)=\sqrt{x+3}$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h+3}-\sqrt{x+3}}{h}$$

$$=\frac{0}{0}$$

$$\frac{\sqrt{x+h+3}-\sqrt{x+3}}{h}\cdot\frac{\sqrt{x+h+3}+\sqrt{x+3}}{\sqrt{x+h+3}+\sqrt{x+3}}$$

$$=\frac{(x+h+3)-(x+3)}{h(\sqrt{x+h+3}+\sqrt{x+3}}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{1}{\sqrt{x+h+3}+\sqrt{x+3}}=\frac{1}{2\sqrt{x+3}}$$

18) Find the equation for the normal line to the graph of $f(x)=5-x^2$ at $x=2$.

$f'(x)=-2x \Rightarrow f'(2)=-2(2)=-4$

$f(2)=5-2(2)^2=5-8=-3$

Slope of normal line is $\frac{1}{4}$

$y+3=\frac{1}{4}(x-2)$ ← normal line

19) Find $f'(1)$, if it exists, when $f(x)=\begin{cases}3x^2 & \text{if } x\le 1\\ 2x^3+1 & x>1\end{cases}$

$$f_-'(1)=\lim_{h\to 0}\frac{3(1+h)^2-3(1)^2}{h}=\lim_{h\to 0}\frac{3(1+2h+h^2)-3}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(6+h)}{\cancel{h}}=\lim_{h\to 0}6+h=6$$

$$f_+'(1)=\lim_{h\to 0}\frac{2(1+h)^3+1-3}{h}$$

$$=\lim_{h\to 0}\frac{2(1+3h+3h^2+h^3)+1-3}{h}=6$$

$f'(1)=6$

20) Determine the values A and B that make $f'(x)$ continuous everywhere.

$$f(x)=\begin{cases}4x & \text{if } x\le 1\\ Ax+B & x>1\end{cases}\Rightarrow f'(x)=\begin{cases}4 & \text{if } x\le 1\\ A & x>1\end{cases}$$

$f'(1)=4$

$$\lim_{x\to 1^-}f'(x)=4=A=\lim_{x\to 1^+}f'(x)\quad\therefore\ \boxed{A=4}$$

21) Find the inverse of $f(x)=(x^3+4)^{1/3}$ if possible.

Suppose $f(x_1)=f(x_2)$. Then $(x_1^3+4)^{1/3}=(x_2^3+4)^{1/3}$ $\Rightarrow$ f is 1-1 and hence invertible

$x_1^3+4=x_2^3+4$

$x_1^3=x_2^3$

$x_1=x_2$

$x=(y^3+4)^{1/3}$

$\Rightarrow x^3=y^3+4$

$\Rightarrow y^3=x^3-4$

$\Rightarrow y=(x^3-4)^{1/3}$

$$\boxed{f^{-1}(x)=(x^3-4)^{1/3}}$$

22) Use the ε-δ defn of a limit to prove that $\lim_{x \to 3}(2x+7) = 13$.

Pf Fix $\varepsilon > 0$. Suppose there exists $\delta > 0$ s.t. if $0 < |x-3| < \delta$, then $|2x+7-13| < \varepsilon$. Well, $|2x-6| = 2|x-3| < 2\delta$ implies we should let $\delta = \frac{\varepsilon}{2}$.

Check: $\forall \varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$. Then if $0 < |x-3| < \delta$, we find that $|(2x+7)-13| = |2x-6| = 2|x-3| < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon$ ∎