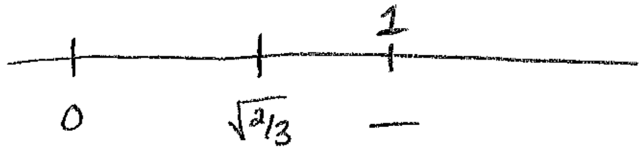
Homework #10 Solutions

Section 11.3

#8) Use the Integral Test to determine whether $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges or diverges.

Let $f(x) = x^2 e^{-x^3} = \frac{x^2}{e^{x^3}}$. Note that exponentials are positive functions. Thus, f is continuous since it is the ratio of continuous functions. Also, f is positive since $x^2 > 0$ for $x \geq 1$. To see that f is decreasing, note that $f'(x) = \frac{2x e^{x^3} - 3x^4 e^{x^3}}{e^{2x^3}} = \frac{x e^{x^3}(2 - 3x^2)}{e^{2x^3}} = \frac{x(2-3x^2)}{e^{x^3}}$

$f'(x) = 0$ if $x = 0$ or $x = \pm\sqrt{2/3}$



$f'(1) = \frac{1(2-3)}{e^1} < 0$

$\Longrightarrow f$ is decreasing for $x \geq 1$

$\therefore$ The integral test will determine the convergence of $\sum n^2 e^{-n^3}$.

$u = x^3 \Rightarrow du = 3x^2 dx$

$$\int_1^{\infty} \frac{x^2}{e^{x^3}}\,dx = \frac{1}{3}\int_1^{\infty} e^{-u}\,du = \frac{1}{3}\lim_{t\to\infty} -e^{-u}\Big|_1^{t^3} = \frac{1}{3}\lim_{t\to\infty}(e^{-1} - e^{-t^3}) = \frac{1}{3e}$$

Hence, $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges by integral test.

#31) For what values of p is $\sum_{n=1}^{\infty} n(1+n^2)^p$ convergent.

If $p > 0$, then $n(1+n^2)^p \to \infty$ as $n \to \infty$. Thus, $\sum n(1+n^2)^p$ diverges by the Basic Divergence test. If $p = 0$, then $\sum n \to \infty$. Suppose $p < 0$. Let $f(x) = x(1+x^2)^p = \frac{x}{(1+x^2)^{|p|}}$. Note that f is continuous and positive since $1+x^2 > 0$ for $x \geq 1$. Also, for $x > 1$ $f'(x) = \frac{1-x^2}{(1+x^2)^{|p|+1}} < 0$ by First Derivative test.

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If $p=-1$, then $\int_1^\infty \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^\infty \frac{1}{u}\,du = \frac{1}{2}\lim_{t\to\infty} \ln u \Big|_2^{1+t^2}$

$$= \frac{1}{2}\lim_{t\to\infty}\left(\ln(1+t^2) - \ln 2\right) \to \infty$$

For $p<0$ s.t. $p \neq -1$, $\int_1^\infty \frac{x}{(1+x^2)^{|p|}}\,dx = \frac{1}{2}\int_1^\infty \frac{1}{u^{|p|}}\,du$

$$= \frac{1}{2(1-|p|)} \lim_{t\to\infty} u^{1-|p|}\Big|_2^{1+t^2}$$

$$= \frac{1}{2(1-|p|)} \lim_{t\to\infty} \left((1+t^2)^{1-|p|} - 2^{1-|p|}\right)$$

If $-1<p<0$, then $\left((1+t^2)^{1-|p|} - 2^{1-|p|}\right) \to \infty$ as $t\to\infty$ so the series will diverge.

If $p<-1$, then $\left((1+t^2)^{1-|p|} - 2^{1-|p|}\right) \to -2^{1-|p|}$ so

$\int_1^\infty \frac{x}{(1+x^2)^{|p|}}\,dx < \infty \Rightarrow \sum n(1+n^2)^p$ converges by integral test.

$\therefore \sum n(1+n^2)^p$ is convergent if $p<-1$.

#40) How many terms of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ would you need to add to find its sum within 0.01?

Note that $f(x) = \frac{1}{x(\ln x)^2}$ is continuous as $x(\ln x)^2$ is, the composition and product of continuous functions. Also, f is positive for $x \geq 2$. Since both x and $\ln x$ are increasing, f is a decreasing function.

$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \int_2^\infty u^{-2}\,du = \lim_{t\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty}\left(\frac{1}{\ln 2} - \frac{1}{\ln t}\right) = \frac{1}{\ln 2} < \infty$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by integral test.

Thus, $R_n \leq \int_n^\infty \frac{1}{x(\ln x)^2}\,dx = \frac{1}{\ln n} < 0.01^{=\frac{1}{100}}$ if $100 < \ln n$ or $e^{100} < n$.

Section 11.4

#12) Determine whether $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ converges or diverges.

$$\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} = \sum_{k=1}^{\infty} \frac{(2k-1)(k+1)}{(k^2+4)^2}$$

Note that $a_n = \frac{(2k-1)(k+1)}{(k^2+4)^2}$ is dominated by how $b_n = \frac{2k^2}{k^4} = \frac{2}{k^2}$ behaves.

$$\lim_{k\to\infty} \frac{(2k-1)(k+1)}{(k^2+4)^2} \cdot \frac{k^2}{2} = \lim_{k\to\infty} \frac{2k^4+k^3-k^2}{2k^4+16k^2+32} = 2.$$

Since $0 < 2 < \infty$, the limit comparison test shows that $\sum \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ behaves like $\sum \frac{2}{k^2}$ which converges by p-series. Hence, $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ converges.

Section 11.5

$n=1 \to a_n = \frac{1}{5} = 0.2$

$n=2 \to a_n = \frac{\sqrt{2}}{7} \approx 0.202$

#10) Determine whether $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$ converges or diverges.

Note that this is an alternating series. Let $f(x) = \frac{\sqrt{x}}{2x+3}$.

$$f'(x) = \frac{(2x+3)(\frac{1}{2}x^{-1/2}) - 2x^{1/2}}{(2x+3)^2} = \frac{\frac{1}{2}x^{-1/2}((2x+3) - 4x)}{(2x+3)} = \frac{-2x+3}{2x^{1/2}(2x+3)}$$

2

$-\frac{3}{2}$ 0 $\frac{3}{2}$ —

$\therefore a_n = \frac{\sqrt{n}}{2n+3} \geq a_{n+1}$ for $n \geq 2$ (1)

(2) $\lim_{n\to\infty} \frac{n^{1/2}}{2n+3} = \frac{\infty}{\infty} \Rightarrow \lim_{n\to\infty} \frac{n^{1/2}}{2n+3} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{4n^{1/2}} = 0$

By (1) and (2), $\sum_{n=2}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$ converges by Alternating series test.

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3} = -\frac{1}{5} + \sum_{n=2}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3} < \infty$ also.

Section 11.6

\# 20) Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$$ converges or diverges.

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)\cancel{(2n)!} \cdot \cancel{(n!)^2}}{(n+1)^2 \cancel{(n!)^2} \cancel{(2n)!}}$

$$= \frac{4n^2+6n+2}{n^2+2n+1} \xrightarrow{n\to\infty} 4 > 1$$

$\therefore \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ diverges by Ratio Test.