Homework #11 Solutions

Section 11.8

#9) Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$.

Let $a_n = \frac{x^n}{n^4 4^n}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{\cancel{n+1}}}{(n+1)^4 4^{\cancel{n+1}}} \cdot \frac{n^4 \cancel{4^n}}{\cancel{|x|^n}} = \frac{|x|}{4} \cdot \left(\frac{n}{n+1}\right)^4 \xrightarrow{n \to \infty} \frac{|x|}{4}$.

By ratio test, the series converges if $\frac{|x|}{4} < 1 \Rightarrow R = 4$ and $\sum_{n=1}^{\infty} \frac{x^n}{n^4 \cdot 4^n}$ converges for $x \in (-4, 4)$.

If $x = 4$, $\sum_{n=1}^{\infty} \frac{1}{n^4} < \infty$ by p-series. $\therefore$ Series converges for $x = 4$.

If $x = -4$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} < \infty$ by alt. series test. $\therefore$ Series converges for $x = -4$. Hence, the interval of convergence is $[-4, 4]$.

#30) Suppose $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence of the following?

Note that the given hypothesis implies that the radius of convergence satisfies $4 \le R < 6$. $\therefore \sum_{n=0}^{\infty} c_n x^n$ converges for $|x| \le 4 \le R < 6$. So,

(a) $\sum c_n$ converges since $1 < 4 \le R$

(b) $\sum c_n 8^n$ diverges since $8 > 6 > R$

(c) $\sum c_n (-3)^n$ converges since $|-3| < 4 \le R$

(d) $\sum c_n (-9)^n$ diverges since $|-9| > 6 > R$.

Section 11.9

#8) Find a power series representation for $f(x) = \frac{x}{2x^2+1}$ and determine the interval of convergence.

$$f(x) = x \cdot \frac{1}{1-(-2x^2)} = x \sum_{n=0}^{\infty} (-2x^2)^n = \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1} |x|^{2n+3}}{2^n |x|^{2n+1}} = 2|x|^2$$

By ratio test, $\sum (-1)^n 2^n x^{2n+1}$ converges if $2|x|^2 < 1 \Rightarrow |x| < \frac{1}{\sqrt{2}}$

$x = \frac{1}{\sqrt{2}}$: $\sum (-1)^n 2^n \cdot (2^{-1/2})^{(2n+1)} = \sum (-1)^n \cdot \frac{1}{\sqrt{2}}$ diverges

$x = -\frac{1}{\sqrt{2}}$: $\sum (-1)^n 2^n (-2^{-1/2})^{(2n+1)} = \sum (-1)^{n+1} \cdot \frac{1}{\sqrt{2}}$ diverges

Interval of convergence: $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

#27) Evaluate $\int x^2 \ln(1+x)\,dx$ as a power series. What is the radius of convergence?

Let $f(x) = \ln(1+x)$. Then $f'(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$. Thus,

$$f(x) = \int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty} (-x)^n\,dx = \sum_{n=0}^{\infty} \int (-x)^n\,dx = C + \sum_{n=0}^{\infty} \frac{-(-x)^{n+1}}{n+1}$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1}$$ which converges for $|x| < 1$.

If $x = 0$, $0 = f(0) = C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{0^{n+1}}{n+1} \Rightarrow C = 0$.

Therefore, $\int x^2 \ln(1+x)\,dx = \int x^2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^{n+2}\,dx$

$$= D + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} \int x^{n+2}\,dx$$

$$= D + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n(n+3)} x^{n+3}$$

Let $a_n = \frac{(-1)^{n-1}}{n(n+3)} x^{n+3}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{\cancel{n+4}}}{(n+1)(n+4)} \cdot \frac{n(n+3)}{\cancel{|x|^{n+3}}} = \frac{n^2+3n}{n^2+5n+4} \cdot |x|$

By ratio test, the series converges iff $|x| \le 1 =: R$.

Section 11.10

#9) Use defn of Taylor series to find the first four nonzero terms of the series for $f(x) = \sin x$ centered at $a = \frac{\pi}{6}$

$$a_n = \frac{f^{(n)}(\pi/6)}{n!} (x - \pi/6)^n \Rightarrow a_0 = \frac{1}{2},\ a_1 = \frac{\sqrt{3}}{2}(x - \pi/6),$$

$$a_2 = -\frac{1}{4}(x - \pi/6)^2,\ a_3 = -\frac{\sqrt{3}}{12}(x - \pi/6)^3$$

#12) Find the McClaurin Series for $f(x) = \ln(1+x)$ and the associated radius of convergence.

Claim: $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ for $n \ge 1$.

pf $n=1$: $f'(x) = \frac{1}{1+x} = \frac{(-1)^0 \cdot 0!}{(1+x)^1}$ ✓

suppose Claim true for some $n \in \mathbb{N}$.

$$f^{(n+1)}(x) = \frac{d}{dx}(f^n(x)) = \frac{d}{dx}\left(\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}\right)$$

$$= (-1)^{n-1}(n-1)! \cdot \frac{d}{dx}((1+x)^{-n}) = \frac{(-1)^n n!}{(1+x)^{n+1}}$$

∴ Claim true for $n \in \mathbb{N}$.

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \ln(1) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{|x|^n}}$

$= \frac{n}{n+1}|x| \to |x|$

By Ratio Test, the series has radius of convergence $R = 1$.