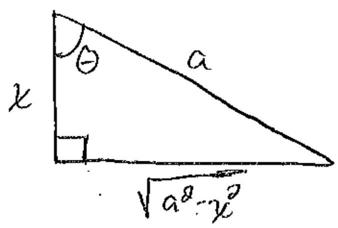
Homework 2 Solutions

Section 7.3

15) Evaluate $\int_0^a x^2\sqrt{a^2-x^2}\,dx$. (Assuming $a \neq 0$)



$a\cos\theta = x$

$-a\sin\theta\,d\theta = dx$

$x^2 = a^2\cos^2\theta$

$\sqrt{a^2-x^2} = a\sin\theta$

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_{\pi/2}^{0} a^2\cos^2\theta \cdot a\sin\theta \cdot -a\sin\theta\,d\theta$$

$$= \int_0^{\pi/2} a^4\cos^2\theta\sin^2\theta\,d\theta$$

$$= a^4\int_0^{\pi/2}\left(\frac{1+\cos(2\theta)}{2}\right)\left(\frac{1-\cos(2\theta)}{2}\right)d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2}\left(1-\cos^2(2\theta)\right)d\theta$$

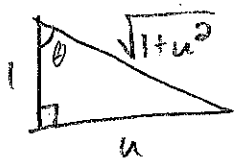
$$= \frac{a^4}{4}\left[\int_0^{\pi/2} d\theta - \frac{1}{2}\int_0^{\pi}\cos^2 u\,du\right]$$

$$= \frac{a^4}{4}\left[\frac{\pi}{2} - \frac{1}{4}\int_0^{\pi}(1+\cos(2u))\,du\right]$$

$$\boxed{\frac{\pi a^4}{16}} = \frac{a^4}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{a^4}{4}\left[\frac{\pi}{2} - \frac{1}{4}\left(u + \frac{1}{2}\sin(2u)\Big|_0^{\pi}\right)\right]$$

30) Evaluate $\int_0^{\pi/2}\frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$.

First, if $u = \sin t$, then $du = \cos t\,dt$. Therefore,



$$\int_0^{\pi/2}\frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt = \int_0^1\frac{du}{\sqrt{1+u^2}} = \int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{\sec\theta}$$

$u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$\sqrt{1+u^2} = \sec\theta$

$y = \sec\theta + \tan\theta$

$dy = (\sec\theta\tan\theta + \sec^2\theta)\,d\theta$

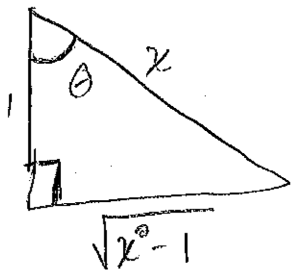
$$= \int_0^{\pi/4}\sec\theta\,d\theta = \int_0^{\pi/4}\sec\theta\,\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$

$$= \int_1^{1+\sqrt{2}}\frac{dy}{y} = \ln y\Big|_1^{1+\sqrt{2}} = \ln(1+\sqrt{2}) - \ln 1$$

$$\boxed{= \ln(1+\sqrt{2})}$$

33) Find the average value of $f(x) = \frac{\sqrt{x^2-1}}{x}$ on $1 \le x \le 7$.

$$\text{Avg}(f, [1,7]) = \frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{6}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx$$



$x = \sec\theta$
$dx = \sec\theta\tan\theta\,d\theta$
$\sqrt{x^2-1} = \tan\theta$

$$= \frac{1}{6}\int \frac{\tan\theta}{\cancel{\sec\theta}}\,\cancel{\sec\theta}\tan\theta\,d\theta$$

$$= \frac{1}{6}\int \tan^2\theta\,d\theta = \frac{1}{6}\int(\sec^2\theta - 1)\,d\theta$$

$$= \frac{1}{6}(\tan\theta - \theta) = \frac{1}{6}\left(\sqrt{x^2-1} - \sec^{-1}x \Big|_1^7\right)$$

$$= \frac{1}{6}\left(\sqrt{48} - \sec^{-1}(7)\right)$$

Section 7.4

3) Write out the partial fraction decomposition of

a) $$\frac{1}{x^2+x^4} = \frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

b) $$\frac{x^3+1}{x^3-3x^2+2x} = 1 + \frac{3x^2-2x+1}{x^3-3x^2+2x} = 1 + \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$\begin{array}{r} 1 \\ x^3-3x^2+2x \,\big)\, x^3 \qquad\quad +1 \\ -(\cancel{x^3}-3x^2+2x) \\ \hline 3x^2-2x+1 \end{array}$$

15) Evaluate $\int_{-1}^{0} \frac{x^3-4x+1}{x^2-3x+2}\,dx$.

$$\begin{array}{r} x+3 \\ x^2-3x+2 \,\big)\, x^3 - 4x + 1 \\ -(\cancel{x^3}+2x \quad -3x^2) \\ \hline 3\cancel{x^2}-6x+1 \\ -(\cancel{3x^2}-9x+6) \\ \hline 3x-5 \end{array}$$

$$\int_{-1}^{0} \frac{x^3-4x+1}{x^2-3x+2}\,dx = \int_{-1}^{0}\left(x+3+\frac{3x-5}{x^2-3x+2}\right)dx$$

$$= \int_{-1}^{0}\left(x+3+\frac{2}{x-1}+\frac{1}{x-2}\right)dx$$

$$= \left(\frac{x^2}{2} + 3x + 2\ln|x-1| + \ln|x-2| \Big|_{-1}^{0}\right)$$

$$= +\cancel{2\ln|-1|} + \ln|-2| - \left(\tfrac{1}{2} - 3 + 2\ln|-2| + \ln|-3|\right)$$

$$= \frac{5}{2} - \ln 2 - \ln 3$$

$$3x-5 = A(x-2)+B(x-1)$$
$$= Ax - 2A + Bx - B$$
$$= (A+B)x - 2A - B$$

$$\begin{array}{r} A+\cancel{B} = 3 \\ -2A-\cancel{B} = -5 \\ \hline -A = -2 \end{array} \qquad \boxed{\begin{array}{l} A = 2 \\ B = 1 \end{array}}$$

30) Evaluate $\displaystyle\int \frac{x^3-2x^2+2x-5}{x^4+4x^2+3}\,dx$. $\quad x^4+4x^2+3=(x^2+3)(x^2+1)$

$$\begin{aligned} x^3-2x^2+2x-5 &= (Ax+B)(x^2+3)+(Cx+D)(x^2+1) \\ &= Ax^3+3Ax+Bx^2+3B+Cx^3+Cx+Dx^2+D \\ &= (A+C)x^3+(B+D)x^2+(3A+C)x+3B+D \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -2 \\ 3 & 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 1 & -5 \end{array}\right)$$

$\Downarrow$

$A=\frac{1}{2}$ $B=\frac{-3}{2}$ $C=\frac{1}{2}$ $D=-\frac{1}{2}$

$$\begin{aligned} \int \frac{x^3-2x^2+2x-5}{x^4+4x^2+3}\,dx &= \int\left(\frac{1}{2}\cdot\frac{x-3}{x^2+1}+\frac{1}{2}\cdot\frac{x-1}{x^2+3}\right)dx \\ &= \frac{1}{2}\int\left(\frac{x}{x^2+1}+\frac{x}{x^2+3}\right)dx-\frac{1}{2}\int\left(\frac{3}{x^2+1}+\frac{1}{x^2+3}\right)dx \\ &= \frac{1}{4}\left(\int \frac{1}{u}\,du+\int\frac{1}{u'}\,du'\right)-\frac{1}{2}\int\left(\frac{3}{x^2+1}+\frac{1}{x^2+3}\right)dx \qquad u=x^2+1,\ u'=x^2+3 \\ &= \frac{1}{4}\left(\ln(x^2+1)+\ln(x^2+3)\right)-\frac{3}{2}\arctan(x)-\frac{1}{2\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)+C \end{aligned}$$

Not Graded

48) $\displaystyle\int \frac{\sin x}{\cos^2 x-3\cos x}\,dx$

Let $u=\cos x$, then $du=-\sin x\,dx$.

$$\int \frac{\sin x}{\cos^2 x-3\cos x}\,dx=\int\frac{du}{u^2-3u}=\frac{-1}{3}\int\left(\frac{1}{u-3}-\frac{1}{u}\right)du$$

$$\begin{aligned} 1 &= A(u-3)+Bu \\ &= (A+B)u-3A \end{aligned}$$

$A+B=0$

$-3A=1 \Rightarrow A=-\frac{1}{3}$

$\Rightarrow B=\frac{1}{3}$

$$=\frac{-1}{3}\left(\ln|u-3|-\ln|u|\right)+C$$

$$=\frac{-1}{3}\left(\ln|\cos x-3|-\ln|\cos x|\right)+C$$