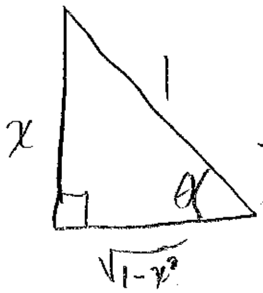
Homework 3 Solutions

Section 7.5

#16) $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cancel{\cos\theta}}\cancel{\cos\theta}\,d\theta$



$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$\sqrt{1-x^2} = \cos\theta$

$x = 0 \Rightarrow \theta = 0$

$x = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$

$$= \frac{1}{2}\int_0^{\pi/4} (1-\cos(2\theta))\,d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/4}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\sin\left(2\cdot\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi - 2}{8}$$

#65) $\displaystyle\int \frac{\sin(2x)}{1+\cos^4(x)}\,dx = 2\int \frac{\sin x\cos x}{1+\cos^4 x}\,dx$ Let $u = \cos x$, $du = -\sin x\,dx$

$\displaystyle = -2\int \frac{u}{1+u^4}\,du$ Now let $w = u^2$, then $dw = 2u\,du$

$\displaystyle = -\int \frac{1}{1+w^2}\,dw = -\arctan w + C$

$= -\arctan(\cos^2 x) + C$

Section 7.7

#22) How large should n be to guarantee that the Simpson's Rule approximation to $\int_0^1 e^{x^2}dx$ is accurate to within 0.00001?

Note that $\frac{d^4}{dx^4}(e^{x^2}) = 4e^{x^2}(4x^4+12x^2+3)$. Since both factors are continuous and increasing, $\left|\frac{d^4}{dx^4}(e^{x^2})\right| \le 76e$ on $(0,1)$.

Then $|E_S| \le \frac{76e(1-0)^5}{180n^4}$. The desired n must satisfy $\frac{19e}{45n^4} < 0.00001 = \frac{1}{100000}$

$\Rightarrow n^4 > \frac{19e\cdot 20000}{9} \approx 114772 \Rightarrow n > 18.4$. Therefore, even $n \ge 20$ is sufficiently large for such accuracy.

§7.5 #41)

$$\int \theta \tan^2\theta \, d\theta = \theta\tan\theta - \theta^2 - \int (\tan\theta - \theta)\, d\theta$$

$u=\theta \quad v=\tan\theta-\theta$

$du = d\theta \quad dv = \tan^2\theta \, d\theta$

$$= \theta\tan\theta - \frac{\theta^2}{2} - \int \frac{\sin\theta}{\cos\theta}\, d\theta$$

$$= \theta\tan\theta - \frac{\theta^2}{2} + \ln|\cos\theta| + C$$

Section 7.8

#30) $$\int_{-1}^{2} \frac{x}{(x+1)^2}\,dx = \lim_{t\to -1^+}\int_t^2 \frac{x}{(x+1)^2}$$

Let $u=x+1$, then $du=dx$ and $x=u-1$,

$x=2 \Rightarrow u=3 \quad x=t \Rightarrow u=t+1$

$$= \lim_{t\to -1^+}\int_{t+1}^{3} \frac{u-1}{u^2}\,du$$

$$= \lim_{t\to -1^+}\int_{t+1}^{3}\left(\frac{1}{u}-\frac{1}{u^2}\right)du = \lim_{t\to -1^+}\left(\ln|u| + \frac{1}{u}\Big|_{t+1}^{3}\right)$$

$$= \ln 3 + \frac{1}{3} - \left(\lim_{t\to -1}\ln(t+1) + \frac{1}{t+1}\right)$$

$$= \ln 3 + \frac{1}{3} + \infty - \infty = \infty \text{ by } ☆$$

$\therefore \int_{-1}^{2} \frac{x}{(x+1)^2}\,dx$ diverges.

$\lim_{x\to 0^+}\left(\ln x + \frac{1}{x}\right) = \infty - \infty$

$= \lim_{x\to 0^+} \frac{x\ln x + 1}{x} = \frac{0\cdot -\infty}{0}$

$\lim_{x\to 0^+} x\ln x + 1 = 0\cdot -\infty$

$\Rightarrow \lim_{x\to 0^+}\frac{\ln x}{1/x} = \frac{-\infty}{\infty}$

$L'H \Rightarrow \lim_{x\to 0^+}\frac{1}{x}\cdot -x^2$

$= \lim_{x\to 0^+} -x = 0$

$\Rightarrow \lim_{x\to 0^+}\frac{x\ln x + 1}{x} \overset{L'H}{=} \lim_{x\to 0^+}\frac{1+\ln x}{1} = -\infty$ ☆

#50) $$\int_1^{\infty} \frac{1+\sin^2 x}{\sqrt{x}}\,dx$$

Note that $0 \le \sin^2 x \le 1 \Rightarrow \frac{1}{\sqrt{x}} \le \frac{1+\sin^2 x}{\sqrt{x}}$.

Thus, $\int_1^{\infty}\frac{1}{\sqrt{x}} \le \int_1^{\infty}\frac{1+\sin^2 x}{\sqrt{x}}\,dx$.

$$\int_1^{\infty}\frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^t x^{-1/2}\,dx = \lim_{t\to\infty} 2\sqrt{x}\,\Big|_1^t = \lim_{t\to\infty}(2\sqrt{t}-2) = \infty$$

Since $\int_1^{\infty}\frac{1}{\sqrt{x}}\,dx$ diverges, so does $\int_1^{\infty}\frac{1+\sin^2 x}{\sqrt{x}}\,dx$.