Homework #5 Solutions

Section 8.4

20) Use Poiseuille's Law to show that if R_0 and P_0 are normal values of the radius and pressure in an artery and the constricted values are R and P, then for the flux to remain constant, P and R are related by $\frac{P}{P_0} = \left(\frac{R_0}{R}\right)^4$. Deduce that if the radius of an artery is reduced to $3/4$ its former value, then the pressure is more than tripled.

For the flux to stay constant, $F_0 = F$, i.e.

$$F_0 = \frac{\pi P_0 R_0^4}{8\eta l} = \frac{\pi P R^4}{8\eta l} = F. \text{ Therefore, } P_0 R_0^4 = P R^4$$

$$\Rightarrow \frac{P}{P_0} = \frac{R_0^4}{R^4} = \left(\frac{R_0}{R}\right)^4.$$

Therefore, if $R = \frac{3}{4}R_0$, then $\frac{P}{P_0} = \left(\frac{R_0}{3/4 R_0}\right)^4 = \left(\frac{4}{3}\right)^4 = \frac{256}{81} \approx 3.16.$

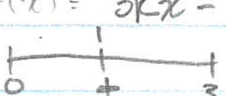
Section 8.5

6) Let $f(x) = k(3x - x^2)$ if $0 \le x \le 3$ and $f(x) = 0$ if $x < 0$ or $x > 3$.

a) For what values of k is f a prob. density func.?

Note we can assume $k \neq 0$ or else $f \equiv 0$ and not a pdf.

$f(x) = 3kx - kx^2 = kx(3-x) \Rightarrow f(x) = 0$ if $x = 0, 3$.



(assuming k is positive)

Thus, $f(x) \geq 0 \ \forall x \in \mathbb{R}$ if k is positive.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^3 k(3x - x^2)\,dx = k\left(\frac{3}{2}x^2 - \frac{x^3}{3}\Big|_0^3\right) = k\left(\frac{27}{2} - 9\right)$$

$$= k\left(\frac{9}{2}\right) \quad \therefore f \text{ is a pdf if } k = \frac{2}{9}.$$

b) For that value of k, find $P(X>1)$.

$$P(X>1) = \int_1^{\infty} f(x)\,dx = \int_1^3 \frac{2}{9}(3x-x^2)\,dx = \frac{2}{9}\left(\frac{3}{2}x^2 - \frac{x^3}{3}\Big|_1^3\right)$$
$$= \frac{2}{9}\left(\frac{27}{2} - 9 - \left(\frac{3}{2} - \frac{1}{3}\right)\right) = \frac{2}{9}\left(12 - \frac{26}{3}\right) = \frac{2}{9}\cdot\frac{10}{3} = \frac{20}{27}$$

c) Find the mean.

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{2}{9}\int_0^3 x(3x-x^2)\,dx = \frac{2}{9}\int_0^3 3x^2 - x^3\,dx$$
$$= \frac{2}{9}\left(x^3 - \frac{x^4}{4}\Big|_0^3\right) = \frac{2}{9}\left(27 - \frac{81}{4}\right) = \frac{2}{9}\left(\frac{27}{4}\right) = \frac{3}{2}$$

$$\begin{array}{r} 27 \\ \times 4 \\ \hline 108 \end{array} \qquad \begin{array}{r} 108 \\ -81 \\ \hline 27 \end{array}$$

Section 9.1

4) a) For what values of k does the function $y = \cos(kt)$ satisfy the differential equation $4y'' = -25y$?

$y' = -k\sin(kt)$, $y'' = -k^2\cos(kt) \Rightarrow -4k^2\cos(kt) = -25\cos(kt)$

or $k^2 = \frac{25}{4}$. $\therefore$ k should equal $\pm\frac{5}{2}$.

b) For those values of k, verify that the family of functions $y = A\sin(kt) + B\cos(kt)$ are also solutions.

$y' = Ak\cos(kt) - Bk\sin(kt)$, $y'' = -Ak^2\sin(kt) - Bk^2\cos(kt)$

$$\Rightarrow 4y'' = -4k^2(A\sin(kt) + B\cos(kt)) = -4\left(\pm\frac{5}{2}\right)^2(A\sin(kt) + B\cos(kt))$$
$$= -25(A\sin(kt) + B\cos(kt))$$
$$= -25y$$

9) A population is modeled by the DE $\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$.

a) For what values of P is the populating increasing?

P increasing if $\frac{dP}{dt} > 0 \ldots$

$\frac{dP}{dt} > 0 \Rightarrow 1 - \frac{P}{4200} > 0 \Rightarrow 0 < P < 4200$

since $1.2P \geq 0$ for all time

b) For what values of P is the population decreasing?

P decreasing if $\frac{dP}{dt} < 0 \ldots$

$\frac{dP}{dt} < 0 \Rightarrow 1 - \frac{P}{4200} < 0 \Rightarrow P > 4200$

c) What are the equilibrium solutions?

P is an equilibrium state if $\frac{dP}{dt} = 0 \ldots$

$1.2P\left(1 - \frac{P}{4200}\right) = 0 \Rightarrow \boxed{P = 0}$ or $1 - \frac{P}{4200} = 0$

$\boxed{P = 4200}$

Section 9.2

3) The direction field for $y' = 2 - y$ is III.

4) The direction field for $y' = x(2-y)$ is I.

5) The direction field for $y' = x + y - 1$ is IV.

6) The direction field for $y' = \sin x \sin y$ is II.

22) Use Euler's method with size step 0.2 to estimate $y(1)$, where $y(x)$ is the solution to $y' = x^2 y - \frac{1}{2}y^2$ w/ I.C. $y(\overset{x_0}{0}) = \overset{y_0}{1}$.

$y_1 = 1 + 0.2(\cancel{0^2 \cdot 1} - \frac{1}{2}(1)^2) = \frac{9}{10}$

$y_2 = \frac{9}{10} + 0.2((0.2)^2(\frac{9}{10}) - \frac{1}{2}(\frac{9}{10})^2) = 0.8262$

$y_3 = 0.8262 + 0.2(0.4^2(0.8262) - \frac{1}{2}(0.8262)^2) = 0.7844$

$y_4 = 0.7844 + 0.2(0.6^2(0.7844 - 0.5(0.7844)^2 = 0.7793$

$y_5 = 0.7793 + 0.2(0.8^2 .7793 - .5(.7793)^2)$

$= 0.8183$

$\boxed{y_5 = 0.8183}$