# Homework 6 Solutions

## Section 9.3

#9) Solve $\frac{dp}{dt} = t^2p - p + t^2 - 1$.

$$\frac{dp}{dt} = p(t^2-1) + t^2 - 1 = (p+1)(t^2-1)$$

$\Rightarrow \frac{dp}{p+1} = (t^2-1)dt$. Integrating both sides gives

$\ln|p+1| = \frac{t^3}{3} - t + c \Rightarrow p+1 = Ae^{(\frac{t^3}{3} - t)}$ where $A = e^c$.

$\therefore p(t) = A e^{(\frac{t^3}{3} - t)} - 1$

#21) Solve the differential equation $y' = x + y$ by make the change of variables $u = x + y$.

$u = x + y \Rightarrow u' = 1 + y'$ or $y' = u' - 1$

So $u' - 1 = u \Rightarrow u' = u + 1 \Rightarrow \frac{u'}{u+1} = 1$

Integrating both sides gives $\ln|u+1| = x + c$

$u + 1 = Ae^x$ where $A = e^c$.

$\therefore y = Ae^x - x - 1$

## Section 9.4

3) Suppose a population develops according to $\frac{dP}{dt} = 0.05P - 0.0005P^2$ where t is measured in weeks.

a) what is the carrying capacity? What is the value of k?

$0.05P - 0.0005P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - \frac{P}{100})$

$\therefore k = 0.05$ and the carrying capacity is 100

b) Where are the slopes close to 0? Where are they large? Increasing / Decreasing?

Slopes are close to 0 close to $P=0$ and $P=100$. Increasing on $0<P<100$; Decreasing on $P>100$. The vertex of this logistic equation is at $P = \frac{-0.05}{2(-0.0005)} = \frac{500}{10} = 50$. $\therefore$ Slope is largest at $P = 50$.

#17) $\alpha$ birthrate $\quad \alpha > \beta$

$\beta$ death rate $\quad P$ population $\quad \frac{dP}{dt} = kP - m$ where $k = \alpha - \beta$.

a) Find the solution of this equation that satisfies the initial condition $P(0) = P_0$.

$$\frac{dP}{dt} = kP - m \Rightarrow \frac{dP}{kP-m} = dt \qquad u = kP - m \quad du = k\,dP$$

$$\int \frac{dP}{kP-m} = \frac{1}{k}\int \frac{1}{u}\,du = \int dt \Rightarrow \frac{1}{k}\ln|u| = t + c$$

$$\ln|u| = kt + c_1$$

$$kP - m = u = Ae^{kt} \qquad \therefore P(t) = \frac{Ae^{kt} + m}{k}$$

$$P_0 = \frac{Ae^{k(0)} + m}{k} = \frac{A+m}{k}$$

$$\Rightarrow A = kP_0 - m$$

$$\boxed{P(t) = \frac{1}{k}\left((kP_0 - m)e^{kt} + m\right)}$$

b) What condition on $m$ will lead to an exponential expansion of the population?

$kP_0 - m > 0$ will result in exponential expansion. $\therefore$ $m$ must be less than $kP_0$; $m < kP_0$

c) What condition on $m$ will result in a constant population? Decline?

Constant if $m = kP_0$; Decline if $m > kP_0$

d) $k = 0.016$, $m = 210{,}000$, $P_0 = 8{,}000{,}000$

$kP_0 = 128{,}000 < 210{,}000 = m \Rightarrow$ the popolation is declining.

Section 9.5

#14) Solve $t\ln t \frac{dr}{dt} + r = te^t$

$$\frac{dr}{dt} + \frac{r}{t\ln t} = \frac{e^t}{\ln t} \qquad e^{\int \frac{dt}{t\ln t}} = e^{\ln|u|} = |u| = |\ln t|$$

$$\ln t \frac{dr}{dt} + \frac{r}{t} = e^t$$

$$\frac{d}{dt}(r\ln t) = e^t$$

$$r\ln t = e^t + c \rightarrow r(t) = \frac{e^t + c}{\ln t}$$

#17) Solve $t\frac{du}{dt} = t^2 + 3u, \quad t > 0, \quad u(2) = 4$

$$\frac{du}{dt} - \frac{3}{t}u = t \qquad e^{\int -\frac{3}{t}dt} = e^{-3\ln t} = t^{-3}$$

$$t^{-3}\frac{du}{dt} - \frac{3}{t^4}u = t^{-2} \longrightarrow \frac{d}{dt}(t^{-3}u) = t^{-2}$$

$$t^{-3}u = -t^{-1} + c$$

$$u(t) = ct^3 - t^2$$

$u(2) = 4 \Rightarrow 4 = 8c - 4 \Rightarrow c = 1$

$$\boxed{u(t) = t^3 - t^2}$$