Homework #3 Key

Section 2.3:

(32) Evaluate the limit, if it exists.

$$\lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{x^2-(x+h)^2}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{-h(2x+h)}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{-(2x+h)}{x^2(x+h)^2} = \frac{-2x}{x^4}$$

$$= \frac{-2}{x^3}$$

(59) If $\lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$, find $\lim_{x\to 1} f(x)$.

Let $g(x) = \frac{f(x)-8}{x-1}$. Then $f(x) = g(x)(x-1)+8$.

Thus, $\lim_{x\to 1} f(x) = \lim_{x\to 1}[g(x)(x-1)+8]$

$$= \lim_{x\to 1}[xg(x)-g(x)+8]$$

$$= (\lim_{x\to 1} x)(\lim_{x\to 1} g(x)) - \lim_{x\to 1} g(x) + \lim_{x\to 1} 8$$

$$= 1\cdot 10 - 10 + 8 = 8$$

Section 2.4

(29) Prove that $\lim_{x\to 2}(x^2-4x+5)=1$ using the ε-δ defn. of limits.

(pf) Fix $\varepsilon>0$. Suppose there is a $\delta>0$ s.t.

$|(x^2-4x+5)-1|<\varepsilon$ whenever $0<|x-2|<\delta$.

Then $|x^2-4x+4| = |x-2|^2 < \delta^2$. Hence, maybe we should let $\delta=\sqrt{\varepsilon}$.

$\forall \varepsilon>0$, let $\delta=\sqrt{\varepsilon}$. If $0<|x-2|<\delta$, then

$|(x^2-4x+5)-1| = |x^2-4x+4| = |x-2|^2 < \delta^2 = (\sqrt{\varepsilon})^2 = \varepsilon$. ∎

(39) If the function f is defined by $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{otherwise,} \end{cases}$ prove that $\lim_{x \to 0} f(x)$ does not exist.

pf) The negation of the ε-δ defn. of a limit is as follows: $\exists\, \varepsilon > 0$ s.t. $\forall\, \delta > 0$, $|f(x) - L| \geq \varepsilon$ whenever $0 < |x - c| < \delta$.

Let $L \in (-\infty, \infty)$; note that since f only takes value 0 or 1, it suffices to consider $L \in \{0, 1\}$. Without loss of generality, assume $L = 1$. (similar proof works if $L = 0$). Let $\varepsilon = \frac{1}{3}$ and $\delta > 0$. If $0 < |x| < \delta$, then $x \in (-\delta, 0) \cup (0, \delta)$. Note that between any two real numbers there are infinitely many irrational numbers. Choose x to be such a number in $(-\delta, 0) \cup (0, \delta)$. Then $|f(x) - L| = |0 - 1| = 1 \geq \varepsilon$. Hence, the limit does not exist.

Section 2.5

(4) From the given graph, state the intervals on which g is continuous.

g is continuous on $[-3, -2), (-2, -1), (-1, 0]$
$(0, 1)$, and $(1, 3]$.

(25) Explain using theorems 4, 5, 7, and 9 why the function is continuous at every number in its domain. State the domain. $f(x) = \frac{2x^2 - x - 1}{x^2+1}$

Let $g(x) = 2x^2 - x - 1$ and $h(x) = \frac{1}{x^2+1}$. Both of these functions are continuous over $\mathbb{R}$ (g is a Thm polynomial, the denominator of h is always positive). By theorem , the product of h and g is continuous over $\mathbb{R}$.