Homework 4 Key

Section 2.5

45) For what values of c is $f(x) = \begin{cases} cx^2+2x & \text{if } x<2 \\ x^3 - cx & x \geq 2 \end{cases}$ continuous on $(-\infty, \infty)$?

$f(2) = (2)^3 - c(2) = 8-2c$

$\lim_{x\to 2^-} f(x) = 4c+4$

$\lim_{x\to 2^+} f(x) = 8-2c$

Thus, $\lim_{x\to 2} f(x)$ exists only for c such that $8-2c = 4c+4$, i.e. $c = \frac{2}{3}$. Since $f(2) = 8-2c$, $f(2) = \lim_{x\to 2} f(x)$ and f is continuous.

55) Use the Intermediate Value Thm to show that there is a root of $e^x = 3-2x$ on $(0,1)$.

$0 = 3-2x-e^x =: f(x)$ Let $g(x) = e^x$ and $h(x) = 2x-3$. $f = h-g$

Both g, h are continuous over $\mathbb{R}$ (exponential and polynomial). Thus, they're continuous over $[0,1]$. Since the difference of continuous functions is continuous, f is continuous on $[0,1]$. $f(0) = 3-0-1 = 2$ $f(1) = 3-2-e = 1-e < 0$

So $f(1) < 0 < f(0)$. Therefore, by IVT there exists a root to f on the interval $(0,1)$.

Section 2.6

18) Find $\lim_{x\to -\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5}$.

$$\lim_{x\to -\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5} = \frac{\infty}{\infty}$$

$$\lim_{x\to -\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to -\infty} \frac{-(4+6/x-2/x^3)}{-(2-4/x^2+5/x^3)} = \frac{-4}{-2} = 2$$

#50) Find the vertical and horizontal asymptotes of $y = \frac{1+x^4}{x^2-x^4}$.

y is a rational function so there is a horizontal asymptote at $y=-1$ since the degrees of the numerator and denominator are equal. We may also see this by observing that

$$\lim_{x\to\infty} \frac{1+x^4}{x^2-x^4} = \lim_{x\to\infty} \frac{\frac{1}{x^4}+1}{\frac{1}{x^2}-1} = \frac{1}{-1} = -1 \text{ and}$$

$$\lim_{x\to-\infty} \frac{1+x^4}{x^2-x^4} = \lim_{x\to-\infty} \frac{\frac{1}{x^4}+1}{\frac{1}{x^2}-1} = \frac{1}{-1} = -1.$$

Note that $1+x^4$ will not factor so y is irreducible.

$x^2-x^4=0 \Rightarrow x^2(1-x^2)=0 \Rightarrow x=0,\pm1$ are the vertical asymptotes.

Section 2.7

$$\frac{2(x+2)-(2x+1)(1)}{(x+2)^2}$$

#8) Find the tangent line to $y = \frac{2x+1}{x+2}$ at the point $(1,1)$.

$$y' = \lim_{x\to1} \frac{\frac{2x+1}{x+2}-1}{x-1} = \lim_{x\to1} \frac{2x+1-x-2}{(x+2)(x-1)} = \lim_{x\to1} \frac{\cancel{x-1}}{(x+2)\cancel{(x-1)}}$$

$$= \lim_{x\to1} \frac{1}{x+2} = \frac{1}{3}$$

tangent line: $y-1=\frac{1}{3}(x-1)$

#60) Determine whether $f'(0)$ exists if $f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & \text{if } x\neq0 \\ 0 & x=0 \end{cases}$.

$$\lim_{x\to0} \frac{x^2\sin(\frac{1}{x})-0}{x} = \lim_{x\to0} x\sin\left(\frac{1}{x}\right)$$

Since $-1\le\sin(\frac{1}{x})\le1$, $-x\le x\sin(\frac{1}{x})\le x$. $\therefore \lim_{x\to0} -x = 0 = \lim_{x\to0} x$

So by squeeze theorem, $\lim_{x\to0} x\sin(\frac{1}{x}) = 0$. Thus, f is differentiable.