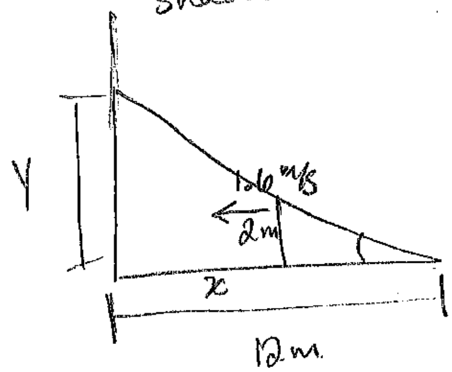
Homework #8 — Selected Solutions

Section 3.9

(#18) A spotlight on the ground shines on a wall 12m away. If a man 2 m tall walks from the spotlight towards the building at 1.6 m/s. How fast is the length of his shadow on the building decreasing when he is 4m away?



let x be the man's distance from the wall.

$$\frac{dx}{dt} = -1.6 \text{ m/s} \qquad \frac{dy}{dt} = ?$$

let y be lenght of man's shadow.

Since these are similar triangles, $\frac{y}{2} = \frac{12}{12-x} \Rightarrow y = \frac{24}{12-x}$

$$\frac{dy}{dt} = \frac{24}{(12-x)^2}\frac{dx}{dt} = \frac{24}{(12-4)^2}\left(-\frac{8}{5}\right) = \boxed{-\frac{3}{5} \text{ m/s}}$$

Section 3.11

(#18) Prove that $\frac{1+\tanh(x)}{1-\tanh(x)} = e^{2x}$.

Recall that $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

Then
$$\frac{1+\tanh(x)}{1-\tanh(x)} = \frac{1+\frac{e^x-e^{-x}}{e^x+e^{-x}}}{1-\frac{e^x-e^{-x}}{e^x+e^{-x}}} = \frac{e^x+\cancel{e^{-x}}+e^x-\cancel{e^{-x}}}{\cancel{e^x}+e^{-x}-(\cancel{e^x}-e^{-x})} = \frac{\cancel{2}e^x}{\cancel{2}e^{-x}} = e^{2x}$$

(#42) Find the derivative of $y = x\tanh^{-1}x + \ln\sqrt{1-x^2}$.

$$y' = \tanh^{-1}x + x\cdot\frac{1}{1-x^2} + \frac{1}{\cancel{2}(1-x^2)}\cdot(-\cancel{2}x)$$

$$= \tanh^{-1}x$$

Section 4.1

#39 Find the critical numbers of $F(x) = x^{4/5}(x-4)^2$.

The domains of $g(x) = x^{4/5}$ and $h(x) = (x-4)^2$ are both $\mathbb{R}$. Therefore, the domain of F is also $\mathbb{R}$.

$$F'(x) = \frac{4}{5}x^{-1/5}(x-4)^2 + 2x^{4/5}(x-4)$$

$$= \frac{2}{5}x^{-1/5}(x-4)\left[2(x-4) + 5x\right] = \frac{2(x-4)(7x-8)}{5x^{1/5}}$$

F' is undefined at $x=0$, $F'(x)=0$ if $x=4$ or $x=\frac{8}{7}$.
$x = 0, 4, \frac{8}{7}$ are in the domain of F and thus are critical numbers.

Section 4.2

#14 Verify that $f(x) = \frac{1}{x}$ satisfies the hypothesis of MVT on $[1,3]$. Then find all numbers c that satisfy the conclusion of MVT.

The domain of f is $D := (-\infty, 0) \cup (0, \infty)$ and f is cont on its domain. $[1,3] \in D \Rightarrow f$ is cont. on $[1,3]$.

$f'(x) = \frac{-1}{x^2}$ which is defined on D. Thus, f is differentiable on $(1,3)$. Hence, the conditions of MVT are satisfied.

There is a $c \in (1,3)$ s.t. $f'(c) = \frac{-1}{c^2} = \frac{f(3)-f(1)}{3-1} = \frac{\frac{1}{3}-1}{2} = \frac{-1}{3}$

$-\frac{1}{c^2} = -\frac{1}{3} \Rightarrow c^2 = 3$ or $c = \pm\sqrt{3}$ but $-\sqrt{3} \notin (1,3)$.

$\therefore$ $\boxed{c = \sqrt{3}}$

#20 Show that $x^3+e^x=0$ has exactly one real root.

Existence

Let $f(x)=x^3+e^x$. Being the sum of continuous functions on $\mathbb{R}$, f is continuous on $\mathbb{R}$. Note that $f(-1)=-1+\frac{1}{e}<0$ and $f(1)=1+e>0$. Thus, by IVT there exists $c\in(-1,1)$ s.t. $f(c)=0$. That is, there is a solution to $x^3+e^x=0$.

Uniqueness

Let c be the solution guaranteed by IVT and d be another distinct solution. Consider the interval $[c,d]$. Note that f is cont. on $[c,d]$. Also note that $f'(x)=3x^2+e^x$, which is differentiable on $\mathbb{R}$. Since $f(c)=f(d)=0$, Rolle's theorem guarantees a $c'\in(c,d)$ such that $f'(c')=0$. However, this is a contradiction since $f'(x)>0$ $\forall x\in\mathbb{R}$. Hence, our assumption that another solution d exists was invalid.

That is, $x^3+e^x=0$ has exactly one real root.