Homework 9 Solutions

Section 4.3

#14 Suppose $f(x) = \cos^2 x - 2\sin x$ on $0 \le x \le 2\pi$.

a) Find the interval on which f is increasing or decreasing.

$$f'(x) = 2\cos x(-\sin x) - 2\cos x$$
$$= -2\cos x(\sin x + 1) = 0$$
$$\cos x = 0 \qquad \sin x = -1 \implies x = \frac{3\pi}{2}, \frac{\pi}{2}$$

$\frac{\pi}{4}$, π, $\frac{7\pi}{4}$

$0 \searrow \frac{\pi}{2} \nearrow \frac{3\pi}{2} \searrow 2\pi$ $\therefore f$ increases on $(\frac{\pi}{2}, \frac{3\pi}{2})$

decreases on $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$

b) Find the local maximum and and minimum values of f.

local max of $f(\frac{3\pi}{2}) = 2$ at $x = \frac{3\pi}{2}$

local min of $f(\frac{\pi}{2}) = -2$ at $x = \frac{\pi}{2}$

by first derivative test in part a).

c) Find the intervals of concavity and the inflection points.

$$f''(x) = -2[(-\sin x)(\sin x + 1) + \cos x \cdot \cos x]$$
$$= -2(\cos^2 x - \sin^2 x - \sin x) = 0$$
$$x = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$\pi/12$, $\pi/4$, π, $7\pi/4$

$0 \searrow \frac{\pi}{6} \nearrow \frac{5\pi}{6} \searrow \frac{3\pi}{2} \searrow 2\pi$

$$\boxed{\text{Points of Inflection } (\tfrac{\pi}{6}, -\tfrac{1}{4}) \; (\tfrac{5\pi}{6}, \tfrac{7}{4})}$$

Concave up: $(\frac{\pi}{6}, \frac{5\pi}{6})$

Concave down: $(0, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, \frac{3\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$

$$f(\tfrac{\pi}{6}) = (\tfrac{\sqrt{3}}{2})^2 - 2(\tfrac{1}{2}) = \tfrac{3}{4} - 1 = -\tfrac{1}{4}$$
$$f(\tfrac{5\pi}{6}) = (\tfrac{\sqrt{3}}{2})^2 - 2(-\tfrac{1}{2}) = \tfrac{3}{4} + 1 = \tfrac{7}{4}$$

Section 4.4

#27 Find the limit of $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{0}{0}$$

By l'Hopital's Rule, $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \frac{0}{0}$.

Applying l'Hopital's Rule again, $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$

#46 Find the limit $\lim_{x\to-\infty} x\ln(1-\frac{1}{x})$.

$\lim_{x\to-\infty} x\ln(1-\frac{1}{x}) = -\infty\cdot\ln(1) = -\infty\cdot 0$

$\Rightarrow$ we must rewrite $x\ln(1-\frac{1}{x})$ somehow.

$\lim_{x\to-\infty} x\ln(1-\frac{1}{x}) = \lim_{x\to-\infty}\frac{\ln(1-\frac{1}{x})}{1/x} = \frac{0}{0}$

$\frac{\frac{1}{1-\frac{1}{x}}\cdot\frac{1}{x^2}}{-\frac{1}{x^2}}$

By l'Hopital's Rule, $\lim_{x\to-\infty} x\ln(1-\frac{1}{x}) = \lim_{x\to-\infty}\frac{-1}{1-1/x} = -1$

Section 4.5

#52 Sketch $y=\frac{\ln x}{x^2}$.

Domain: $(0,\infty)$

Intercepts: $(1,0)$

V.A.: $x=0$

H.A.: $y=0$

Holes: None

Critical #'s: $x=e^{1/2}$

local min/max: max of $\frac{1}{2e}$ at $x=e^{1/2}$

$\frac{5}{6}\cdot\frac{3}{2}$ Interval inc/dec: inc on $(0,e^{1/2})$, dec on $(e^{1/2},\infty)$

POI: $(e^{5/6}, \frac{5}{6e^{5/3}})$ concave up

Intervals of concavity:

Concave up on $(e^{5/6},\infty)$

concave down on $(0,e^{5/6})$

$\lim_{x\to\infty}\frac{\ln x}{x^2} = \frac{\infty}{\infty}$; $\lim_{x\to\infty}\frac{\ln x}{x^2} = \lim_{x\to\infty}\frac{1}{2x^2} = 0$

$\lim_{x\to-\infty}\frac{\ln x}{x^2} = \frac{-\infty}{\infty}$; $\lim_{x\to-\infty}\frac{\ln(x)}{x^2} = \lim_{x\to-\infty}\frac{1}{2x^2} = 0$

$y' = \frac{x^2\cdot\frac{1}{x} - \ln x\cdot 2x}{x^4} = \frac{1-2\ln x}{x^3} = 0$

$1-2\ln x = 0$; $\ln x = \frac{1}{2} \Rightarrow x = e^{1/2}$

$\frac{1}{2}$ $e^{1/2}$ 2

$\frac{1}{2}$, $\frac{1}{4}$

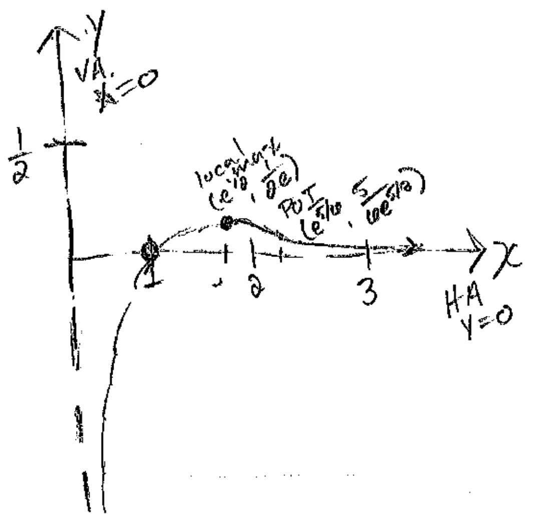
$y'' = \frac{x^3(-\frac{2}{x}) - 3x^2(1-2\ln x)}{x^6} = \frac{-2-3(1-2\ln x)}{x^4}$

$-2-3(1-2\ln x) = 0$

$1-2\ln x = \frac{-2}{3}$; $-2\ln x = -\frac{5}{3}$

$\ln x = \frac{5}{6}$; $x = e^{5/6}$

1 $e^{5/6}$ 10



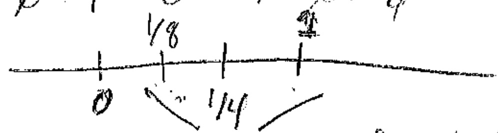
$\lim_{x\to0^+}\frac{\ln x}{x^2} = \frac{-\infty}{0} = -\infty$

Section 4.7

\# 6 What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?

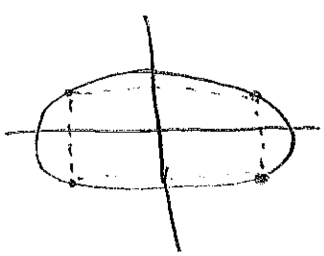
Note that $y_1 = x^2 + 1$ is always greater that $y_2 = x - x^2$. Denote the vertical distance between y_1 and y_2 by
$d(x) = x^2 + 1 - (x - x^2) = 2x^2 - x + 1$.

$d'(x) = 4x - 1 = 0 \Rightarrow x = \frac{1}{4}$



d achieves a minimum of $d(\frac{1}{4}) = 2(\frac{1}{4})^2 - \frac{1}{4} + 1 = \frac{7}{8}$ at $x = \frac{1}{4}$.

\# 26 Find the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



$A = (2x)(2y) = 4xy$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow y^2 = b^2(1 - \frac{x^2}{a^2})$

$\Rightarrow y = b\sqrt{1 - x^2/a^2}$

$A(x) = 4x(b\sqrt{1 - \frac{x^2}{a^2}})$

$A'(x) = 4b\sqrt{1 - \frac{x^2}{a^2}} + 4bx(\cancel{\frac{1}{2}}(1 - \frac{x^2}{a^2})^{-1/2}(-\frac{\cancel{2}x}{a^2})$

$= 4b(1 - \frac{x^2}{a^2})^{-1/2}\left[1 - \frac{x^2}{a^2} - \frac{x^2}{a^2}\right]$

$= \frac{4b(1 - 2\frac{x^2}{a^2})}{\sqrt{1 - \frac{x^2}{a^2}}} \cdot \frac{a}{a} = \frac{4b(a - \frac{2x^2}{a})}{\sqrt{a^2 - x^2}}$

$= \frac{4b}{a} \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} = 0$

$\Rightarrow a^2 - 2x^2 = 0$

$x^2 = \frac{a^2}{2}$ or $x = \frac{a}{\sqrt{2}}$

$A(\frac{a}{\sqrt{2}}) = 4(\frac{a}{\sqrt{2}})b\sqrt{1 - \frac{a^2/2}{a^2}}$

$= \frac{4ab}{\sqrt{2}} \cdot \sqrt{\frac{1}{2}} = \boxed{2ab}$