THE $BMO^{-1}$ SPACE AND ITS APPLICATION TO SCHECHTER’ S INEQUALITY

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Abstract. In this work, we study quadratic form inequalities of Schechter type; i.e., we characterize $f$ for which there exists a positive constant $C$ such that, for every $\epsilon \in (0, \infty)$,

$$\left| \int |u|^2 f dx \right| \leq \epsilon \| \nabla u \|^2_{L^2(\mathbb{R}^d)} + C\epsilon^{-\beta} \| u \|^2_{L^2(\mathbb{R}^d)}, \quad u \in C^\infty_0(\mathbb{R}^d), \quad 0 < \beta < 1$$

Such quadratic form inequalities can be understood entirely in the framework of $BMO^{-1}$, using mean oscillations of $\nabla \Delta^{-1} f$ on balls. We show that this inequality holds if and only if $f \in BMO^{-1}(\mathbb{R}^d)$ if $\beta = 1$ or respectively if $f$ lies in the homogeneous Besov space $\cdot B^{-2\beta}_1 \cdot \infty$ if $0 < \beta < 1$.

1. Introduction

In this paper, we characterize the class of potentials $f \in D'(\mathbb{R}^d)$ such that the quadratic form $\langle f., . \rangle$ has zero relative bound with respect to $H_0 = -\Delta$ on $L^2(\mathbb{R}^d)$ (see [8], X.17). In other words, for $f(x) \geq 0$ in $L^1_{\text{loc}}(\mathbb{R}^d)$, this property can be expressed in the form of the integral inequality:

$$\left| \int |u|^2 f dx \right| \leq \epsilon \| \nabla u \|^2_{L^2(\mathbb{R}^d)} + C\epsilon \| u \|^2_{L^2(\mathbb{R}^d)}, \quad \forall u \in C^\infty_0(\mathbb{R}^d),$$

for all arbitrarily small $\epsilon > 0$ and some $C_\epsilon > 0$. This provides a complete solution to the problem of the infinitesimal form boundedness of the potential energy operator $f$ with respect to the Laplacian $-\Delta$, which is fundamental to quantum mechanics. Its abstract version appears in the so-called KLMN Theorem ([8], Theorem X.17), which is discussed in detail, together with applications to quantum-mechanical Hamiltonian operators and has become an indispensable tool in PDE theory ([7], chap. 5).
Previously, the case of nonnegative $f$ in (1.1) has been studied in a comprehensive way (see e.g. [4], [6], [9], [10]) where different analytic conditions for the so-called trace inequalities of this type can be found.

It is worthwhile to observe that the usual approach is to decompose $f$ into its positive and negative parts: $f = f_+ - f_-$, and to apply the just mentioned results to both $f_+$ and $f_-$. However, this procedure drastically diminishes the class of admissible weights $f$ by ignoring a possible cancellation between $f_+$ and $f_-$. This cancellation phenomenon is evident for strongly oscillating weights considered below. See for example [11].

One of the main results, we prove that inequality (1.1) is equivalent to the existence of $C > 0$ such that

$$\text{(1.2)} \quad |<fu, u>| \leq C R^{2\alpha \beta} \|\nabla u\|^2_{L^2(\mathbb{R}^d)}, \quad \forall u \in C_0^\infty (B(x_0, R))$$

for all ball $B(x_0, R)$. $B(x_0, R)$ is a Euclidean ball of radius $R$ and centered at $x_0$.

Here the "indefinite weight" $f$ may change sign, or even be a complex-valued distribution on $\mathbb{R}^d$, $d \geq 3$. (In the latter case, the left-hand side (1.1) is understood as $|<fu, u>|$, where $<f, u>$ is the quadratic form associated with the corresponding multiplication operator $f$).

We set

$$m_B(g) = \frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} g(y) dy$$

for a ball $B(x_0, R) \subset \mathbb{R}^d$, and denote by $BMO(\mathbb{R}^d)$ the class of $f \in L^1_{loc}(\mathbb{R}^d)$ for which

$$\sup_{R > 0, x_0 \in \mathbb{R}^d} \sup_{B(x_0, R)} \frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} |g(y) - m_B(x_0, R) (g)|^q dy < +\infty,$$

for any $1 \leq q < \infty$.

Now, we characterize the class of potentials $f \in \mathcal{D}'(\mathbb{R}^d)$ which are there exists $C > 0$ such that (1.2) holds for every ball $B(x_0, R)$.

**Theorem 1.** Let $f \in \mathcal{D}'(\mathbb{R}^d)$, $d \geq 2$ and $0 < \beta \leq 1$. Then the following statements are equivalent.

1. There exists a positive constant $C$ such that, for every $\epsilon > 0$,

$$\text{(1.3)} \quad |<fu, u>| \leq \epsilon \|\nabla u\|^2_{L^2(\mathbb{R}^d)} + C \epsilon^{-\beta} \|u\|^2_{L^2(\mathbb{R}^d)}, \quad \forall u \in C_0^\infty (\mathbb{R}^d).$$

2. There exists a positive constant $C$ such that, for every $R > 0$,

$$\text{(1.4)} \quad |<fu, u>| \leq C R^{2\alpha \beta} \|\nabla u\|^2_{L^2(\mathbb{R}^d)}, \quad \forall u \in C_0^\infty (B(x_0, R)).$$
where $C$ does not depend on $x_0$ and $R$.

Define the vector-field $\overrightarrow{F} \in D'(\mathbb{R}^d)^d$ by
\begin{equation}
\langle \overrightarrow{F}, \overrightarrow{\phi} \rangle = - \langle f, \Delta^{-1} \text{div} \overrightarrow{\phi} \rangle,
\end{equation}
for every $\overrightarrow{\phi} = (\phi_1, ..., \phi_d)$ be an arbitrary vector-field in $D \otimes \mathbb{C}^d$. In particular,
\begin{equation}
\langle \overrightarrow{F}, \nabla \psi \rangle = - \langle f, \psi \rangle, \quad \psi \in D(\mathbb{R}^d),
\end{equation}
i.e.,
\begin{equation}
f = \text{div} \overrightarrow{F} \text{ in } D'(\mathbb{R}^d).
\end{equation}

We have to check that the right-hand side of (1.5) is well-defined, which a priori is not obvious. For $\overrightarrow{\phi} \in D \otimes \mathbb{C}^d$, let
\[ w = \Delta^{-1} \text{div} \overrightarrow{\phi}, \]
where $-\Delta^{-1} g = I_2 g$ is the Newtonian potential of $g \in D$. Clearly,
\[ w(x) = O \left( |x|^{1-d} \right) \text{ and } |\nabla w(x)| = O \left( |x|^{-d} \right) \text{ as } |x| \to \infty, \]
and hence
\[ w = \Delta^{-1} \text{div} \overrightarrow{\phi} = -I_2 \text{div} \overrightarrow{\phi} \in H^1(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d). \]

**Remark 1.** When $f(x) \geq 0$ is locally integrable nonnegative function, Theorem 1 makes it possible to reduce the problem of boundedness for general "indefinite" $f$ to the case of nonnegative weights $|\overrightarrow{F}|^2$, which is by now well understood. In particular, combining Theorem 1 and the known criteria in the case $f \geq 0$ (see [4], [6], [9]) we arrive at the following corollary.

**Corollary 1.** Under the assumptions of Theorem 1, the following statements are equivalent.

(i): Inequality (1.3) holds.

(ii): Suppose that $f$ is represented in the form
\begin{equation}
f = \text{div} \overrightarrow{F},
\end{equation}
where $\overrightarrow{F} = \nabla \Delta^{-1} f \in L^2_{\text{loc}}(\mathbb{R}^d)^d$ and the measure $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ defined by
\begin{equation}
d\mu = |\overrightarrow{F}(x)|^2 \, dx
\end{equation}
has the property that, there exists $C > 0$ such that

$$
(1.10) \quad \left| \int |u(x)|^2 \, d\mu \right| \leq \epsilon \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + C e^{-\beta} \|u\|_{L^2(\mathbb{R}^d)}^2, \quad \forall u \in C_0^\infty(\mathbb{R}^d)
$$

for every $\epsilon > 0$.

(iii): For $\mu$ defined by (1.9),

$$
\lim_{R \to 0^+} \sup_{x_0 \in \mathbb{R}^d} \left\{ \frac{\|\mu_B(x_0, R)\|_{L^1(\mathbb{R}^d)}}{\mu(B(x_0, R))} \right\} = 0,
$$

where $\mu_B(x_0, R)$ is the restriction of $\mu$ to the ball $B(x_0, R)$.

Before proceeding to our main result, it is instructive to demonstrate the cancellation phenomenon mentioned above by considering an example of a strongly oscillating weight.

**Example 1.** Let us set

$$
f(x) = |x|^{d-2} \sin \left( \frac{|x|^d}{2} \right),
$$

where $d \geq 3$ is an integer, which may be arbitrary large. Obviously, both $f_+$ and $f_-$ fail to satisfy (1.3) due to the growth of the amplitude at infinity. However,

$$
(1.11) \quad f(x) = \text{div} \overline{F}(x) + O \left( |x|^{-2} \right), \quad \text{where} \quad \overline{F}(x) = -\frac{\overline{F}}{d|x|^2} \cos \left( \frac{|x|^d}{2} \right).
$$

By Hardy’s inequality in $\mathbb{R}^d$, $d \geq 3$,

$$
\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} \, dx \leq \frac{4}{(d-2)^2} \|\nabla u\|_{L^2}^2, \quad u \in C_0^\infty(\mathbb{R}^d),
$$

and hence the term $O \left( |x|^{-2} \right)$ in (1.11) is harmless, whereas $\overline{F}$ clearly satisfies (1.10) since $|\overline{F}(x)|^2 \leq |x|^{-2}$. This shows that $f$ is admissible for (1.2), while $|f|$ is obviously not (see [6]).

**Theorem 2.** Let $f$ be a complex-valued distribution on $\mathbb{R}^d$, $d \geq 3$ and let $0 < \beta \leq 1$. Then (1.4) holds if and only if $f$ is the divergence of a vector-field $\overline{F} : \mathbb{R}^d \to \mathbb{C}^d$ such that

$$
(1.12) \quad \int_{B(x_0, R)} \left| \overline{F}(x) - m_{B(x_0, R)}(\overline{F}) \right|^2 \, dx \leq \text{const} \, R^{d-2+\frac{1}{1+\beta}}, \quad \text{for all} \ R > 0.
$$

where the constant is independent of $x_0$ and $R$. The vector-field $\overline{F} \in L^2_{\text{loc}}(\mathbb{R}^d)$ can be chosen as $\overline{F} = \nabla \Delta^{-1} f$ (see [6]).
Remark 2.

1.: In case, $\beta = 1$, it follows that (1.12) holds if and only if $\nabla F \in BMO (\mathbb{R}^d)$. 
In order, $f \in BMO^{-1} (\mathbb{R}^d) = \mathcal{F}^{-1,2}_{\infty} (\mathbb{R}^d)$, where $\mathcal{F}^{-r,p}$ stands for the scale of homogeneous Triebel-Lizorkin spaces (see [13]). Similarly, in the case $0 < \beta < 1$, (1.12) holds if and only if $\nabla F$ is H"{o}lder-continuous:

$$|\nabla F(x) - \nabla F(y)| \leq c|x - y|^{1+\beta}, \quad |x - y| < R.$$ 

2.: In the case $\beta = 1$, statement (i) of Theorem 2 (sufficiency of the condition $\nabla F \in BMO (\mathbb{R}^d)$) is equivalent via the $H^1 - BMO$ duality to the inequality

$$\|u\|_{H^1(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^d)} \|\nabla F\|_{L^2(\mathbb{R}^d)} , \quad \forall u \in C_0^{\infty} (\mathbb{R}^d).$$ 

Here $H^1 (\mathbb{R}^d)$ is the real Hardy space on $\mathbb{R}^d$ [12]. The preceding estimate yields the following vector-valued inequality which is used in studies of the Navier-Stokes equation, and is related to the compensated compactness phenomenon (see [1]):

$$\|\nabla \nabla u\|_{H^1(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^d)} \|\nabla F\|_{L^2(\mathbb{R}^d)} , \quad \forall u \in C_0^{\infty} (\mathbb{R}^d).$$

Before proving the theorem, let us established certain localized versions of the necessary condition for (1.4). Set

$$\omega_{R,x_0}(x) = \omega \left( \frac{x - x_0}{R} \right)$$

where $\omega \in C_0^{\infty} (B(0,1))$ is a smooth cut-off function with the following properties

$$|\omega(x)| \leq 1 \text{ and } |\nabla \omega(x)| \leq 1 \text{ for } x \in B(0,1).$$

With this definition, we obtain the following more useful statement.

**Proposition 1.** Suppose $f \in D' (\mathbb{R}^d)$ and $0 < \beta \leq 1$. Suppose that (1.4) holds for every $R \in (0, +\infty)$. Let $F$ be defined by $F = \nabla \Delta^{-1} f$.

(a): For $d \geq 3$,

$$\int_{\mathbb{R}^d} |\nabla \Delta^{-1} (\omega_{R,x_0} f)|^2 \, dx \leq CR^{d-2+\frac{2}{\beta}} , \quad 0 < R < +\infty$$
(b): For $d \geq 2$,

$$
\int_{B(x_0, R)} \left| \nabla \Delta^{-1} \left( \omega_{R, x_0} f \right) \right|^2 \, dx \leq C R^{d-2+\frac{1}{1+\beta}} , \quad 0 < R < +\infty
$$

Now we can state the following

**Lemma 1.** Suppose $f \in D' (\mathbb{R}^d)$, $d \geq 2$ and $0 < \beta \leq 1$. Suppose that (1.4) holds for every $R \in (0, +\infty)$. Then we have

$$
\int_{B(x_0, R)} \left| \nabla \Delta^{-1} f - m_{B(x_0, R)} \left( \nabla \Delta^{-1} f \right) \right|^2 \, dx \leq C R^{d-2+\frac{1}{1+\beta}} .
$$

We are now in a position to give the proof of theorem 2. We need only to prove the statement (i) since (ii) follow from Proposition 1 and Lemma 1.

**Proof.** Suppose that $f$ is represented in the form (1.7) so that (1.12) is satisfied for all $R > 0$. Applying the multiplicative inequality nonnegative measures ([5], th.1.4.7) to $\left| \nabla \Delta^{-1} f - m_{B(x_0, R)} \left( \nabla \Delta^{-1} f \right) \right|^2 \, dx$, we get:

$$
\int_{B(x_0, R)} \left| \nabla \Delta^{-1} f - m_{B(x_0, R)} \left( \nabla \Delta^{-1} f \right) \right|^2 \, dx \leq C \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)}^{\frac{2}{1+\beta}} \left\| u \right\|_{L^2(\mathbb{R}^d)}^{\frac{1}{1+\beta}} .
$$

Hence,

$$
| \langle fu, u \rangle | = \left| \left\langle \nabla \Delta^{-1} f - m_{B(x_0, R)} \left( \nabla \Delta^{-1} f \right), \nabla u \right\rangle \right| \leq \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)} \left\| u \right\|_{L^2(\mathbb{R}^d)} \leq C \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)} \left\| u \right\|_{L^2(\mathbb{R}^d)}^{\frac{1}{1+\beta}} .
$$

Combining the preceding estimates with the following inequality ([7], th 3.2.1):

$$
\| u \|_{L^2} \leq C(d) R \| \nabla u \|_{L^2} , \quad u \in C_0^\infty (B(x_0, R)) ,
$$

we get

$$
| \langle fu, u \rangle | \leq C R^{\frac{1}{1+\beta}} \| \nabla u \|_{L^2}^2 , \quad u \in C_0^\infty (B(x_0, R)) .
$$

The proof of theorem 2 is complete. □

We use know characterizations of the Morrey-Campanato spaces. In particular,

**Proposition 2.** For $0 < \beta < 1$, condition (1.12) is equivalent to the condition $\mathcal{F} \in \Lambda_\gamma (\mathbb{R}^d)$ where $\gamma = \frac{1-\beta}{1+\beta}$. In the case $\beta = 1$, we have $\mathcal{F} \in \text{BMO} (\mathbb{R}^d)^d$. 

It is easy to see that in the case $\beta = 1$, the sufficiently part of Theorem 2 is equivalent to inequality:
\[
| \langle F u, \nabla u \rangle | \leq C \left\| F \right\|_{BMO(\mathbb{R}^d)} \| u \|_{L^2(\mathbb{R}^d)} \| \nabla u \|_{L^2(\mathbb{R}^d)}, \forall u \in C_0^\infty (\mathbb{R}^d)
\]

By duality, the preceding inequality yields:
\[
\| u \nabla u \|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C \| u \|_{L^2(\mathbb{R}^d)} \| \nabla u \|_{L^2(\mathbb{R}^d)}, \forall u \in C_0^\infty (\mathbb{R}^d).
\]

where $\mathcal{H}^1(\mathbb{R}^d)$ is a real Hardy space [12]. Such inequalities are useful in hydrodynamics [1]. As an immediate consequence, we obtain the vector-valued quadratic form:
\[
\left\| \left( \nabla u, -u \right) \right\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)} \| \nabla u \|_{L^2(\mathbb{R}^d), d} \|
\]

Both of the preceding inequalities are corollaries of the homogeneous version of the "$\text{div} - \text{curl}$" Lemma [1]. The following corollary which is an immediate consequence of Theorem 2 and the characterizations of Morrey-Campanato spaces [3], gives a necessary and sufficient condition for (1.12) in terms of homogeneous Besov spaces of negative order.

**Corollary 2.** Under the assumptions of Theorem 2, in the case $\beta = 1$, condition (1.12) is equivalent to $f \in BMO^{-1}(\mathbb{R}^d)$. Similarly, in the case $0 < \beta < 1$,
condition (1.12) is equivalent to $f \in B_{\frac{\beta}{1+\beta}, \infty}^{\frac{d}{\beta}}(\mathbb{R}^d)$.

**References**


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