Houston Journal of Mathematics

Volume 33, No. 4, 2007

THE BMO^{-1} SPACE AND ITS APPLICATION TO SCHECHTER' S INEQUALITY

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Communicated by Stephen W. Semmes

Abstract. In this work, we study quadratic form inequalities of Schechter type; i.e., we characterize f for which there exists a positive constant C such that, for every $\epsilon \in (0, \infty)$,

$$\left| \int |u|^2 f dx \right| \leq \epsilon \|\nabla u\|^2_{L^2(\mathbb{R}^d)} + C\epsilon^{-\beta} \|u\|^2_{L^2(\mathbb{R}^d)}, \; u \in C_0^\infty\left(\mathbb{R}^d\right), \; 0 < \beta < 1$$

Such quadratic form inequalities can be understood entirely in the framework of BMO^{-1}, using mean oscillations of $\nabla \Delta^{-1} f$ on balls. We show that this inequality holds if and only if $f \in BMO^{-1}\left(\mathbb{R}^d\right)$ if $\beta = 1$ or respectively if f lies in the homogeneous Besov space $\dot{B}_\infty^{-\frac{2\beta}{1+\beta},\infty}$ if $0 < \beta < 1$.

1. Introduction

In this paper, we characterize the class of potentials $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$ such that the quadratic form $\langle f.,.\rangle$ has zero relative bound with respect to $H_0 = -\Delta$ on $L^2\left(\mathbb{R}^d\right)$ (see [8], X.17). In other words, for $f(x) \geq 0$ in $L^1_{loc}\left(\mathbb{R}^d\right)$, this property can be expressed in the form of the integral inequality :

$$\left| \int |u|^2 f dx \right| \leq \epsilon \|\nabla u\|^2_{L^2(\mathbb{R}^d)} + C_\epsilon \|u\|^2_{L^2(\mathbb{R}^d)}, \; \forall u \in C_0^\infty\left(\mathbb{R}^d\right), \tag{1.1}$$

for all arbitrarily small $\epsilon > 0$ and some $C_\epsilon > 0$. This provides a complete solution to the problem of the infinitesimal form boundedness of the potential energy operator f with respect to the Laplacian $-\Delta$, which is fundamental to quantum mechanics. Its abstract version appears in the so-called KLMN Theorem ([8], Theorem X.17), which is discussed in detail, together with applications to quantum-mechanical Hamiltonian operators and has become an indispensable tool in PDE theory ([7], chap. 5).

Previously, the case of nonnegative f in (1.1) has been studied in a comprehensive way (see e.g. [4], [6], [9], [10]) where different analytic conditions for the so-called trace inequalities of this type can be found.

It is worthwhile to observe that the usual approach is to decompose f into its positive and negative parts : $f = f_+ - f_-$, and to apply the just mentioned results to both f_+ and f_- [6]. However, this procedure drastically diminishes the class of admissible weights f by ignoring a possible cancellation between f_+ and f_-. This cancellation phenomenon is evident for strongly oscillating weights considered below. See for example [11].

One of the main results, we prove that inequality (1.1) is equivalent to the existence of $C > 0$ such that

$$|< fu, u >| \leq C \ R^{\frac{2}{1+\beta}} \left\|\nabla u\right\|^2_{L^2(\mathbb{R}^d)}, \ \ \forall u \in C_0^\infty \left(B\left(x_0, R\right)\right) \tag{1.2}$$

for all ball $B\left(x_0, R\right)$. $B\left(x_0, R\right)$ is a Euclidean ball of radius R and centered at x_0.

Here the "indefinite weight" f may change sign, or even be a complex-valued distribution on $\mathbb{R}^d$, $d \geq 3$. (In the latter case, the left-hand side (1.1) is understood as $|< fu, u >|$, where $< f.,. >$ is the quadratic form associated with the corresponding multiplication operator f).

We set

$$m_B\left(g\right) = \frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} g(y)dy$$

for a ball $B\left(x_0, R\right) \subset \mathbb{R}^d$, and denote by $BMO\left(\mathbb{R}^d\right)$ the class of $f \in L^q_{loc}\left(\mathbb{R}^d\right)$ for which

$$\sup_{R>0} \sup_{x_0 \in \mathbb{R}^d} \frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} \left|g(y) - m_{B(x_0,R)}\left(g\right)\right|^q dy < +\infty,$$

for any $1 \leq q < \infty$.

Now, we characterize the class of potentials $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$ which are there exists $C > 0$ such that (1.2) holds for every ball $B\left(x_0, R\right)$.

Theorem 1. *Let $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$, $d \geq 2$ and $0 < \beta \leq 1$. Then the following statements are equivalent.*

(1) *There exists a positive constant C such that, for every $\epsilon > 0$,*

$$|< fu, u >| \leq \epsilon \left\|\nabla u\right\|^2_{L^2(\mathbb{R}^d)} + C\epsilon^{-\beta} \left\|u\right\|^2_{L^2(\mathbb{R}^d)}, \quad \textit{for all } u \in C_0^\infty\left(\mathbb{R}^d\right). \tag{1.3}$$

(2) *There exists a positive constant C such that, for every $R > 0$,*

$$|< fu, u >| \leq C \ R^{\frac{2}{1+\beta}} \left\|\nabla u\right\|^2_{L^2(\mathbb{R}^d)}, \qquad \textit{for all } u \in C_0^\infty\left(B\left(x_0, R\right)\right) \tag{1.4}$$

where C does not depend on x_0 and R.

Define the vector-field $\overrightarrow{F} \in \mathcal{D}'\left(\mathbb{R}^d\right)^d$ by

$$\left\langle \overrightarrow{F}, \overrightarrow{\phi} \right\rangle = -\left\langle f, \Delta^{-1}\mathrm{div}\,\overrightarrow{\phi} \right\rangle, \tag{1.5}$$

for every $\overrightarrow{\phi} = (\phi_1, ..., \phi_d)$ be an arbitrary vector-field in $\mathcal{D} \otimes \mathbb{C}^d$. In particular,

$$\left\langle \overrightarrow{F}, \nabla\psi \right\rangle = -\left\langle f, \psi \right\rangle, \qquad \psi \in \mathcal{D}\left(\mathbb{R}^d\right), \tag{1.6}$$

i.e.,

$$f = \mathrm{div}\ \overrightarrow{F} \quad \text{in } \mathcal{D}'\left(\mathbb{R}^d\right). \tag{1.7}$$

We have to check that the right-hand side of (1.5) is well-defined, which a priori is not obvious. For $\overrightarrow{\phi} \in \mathcal{D} \otimes \mathbb{C}^d$, let

$$w = \Delta^{-1}\mathrm{div}\,\overrightarrow{\phi},$$

where $-\Delta^{-1}g = I_2 g$ is the Newtonian potential of $g \in \mathcal{D}$. Clearly,

$$w(x) = O\left(|x|^{1-d}\right) \quad \text{and} \quad |\nabla w(x)| = O\left(|x|^{-d}\right) \quad \text{as} \quad |x| \to \infty,$$

and hence

$$w = \Delta^{-1}\mathrm{div}\ \overrightarrow{\phi} = -I_2\mathrm{div}\ \overrightarrow{\phi} \in \dot{H}^1\left(\mathbb{R}^d\right) \cap C^\infty\left(\mathbb{R}^d\right).$$

Remark 1. *When $f(x) \geq 0$ is locally integrable nonnegative function, Theorem 1 makes it possible to reduce the problem of boundedness for general "indefinite" f to the case of nonnegative weights $\left|\overrightarrow{F}\right|^2$, which is by now well understood. In particular, combining Theorem 1 and the known criteria in the case $f \geq 0$ (see* [4], [6], [9]*) we arrive at the following corollary.*

Corollary 1. *Under the assumptions of Theorem 1, the following statements are equivalent.*

- **(i):** *Inequality (1.3) holds.*
- **(ii):** *Suppose that f is represented in the form*

$$f = div\overrightarrow{F}, \tag{1.8}$$

where $\overrightarrow{F} = \nabla\Delta^{-1}f \in L^2_{loc}\left(\mathbb{R}^d\right)^d$ and the measure $\mu \in \mathcal{M}^+\left(\mathbb{R}^d\right)$ defined by

$$d\mu = \left|\overrightarrow{F}(x)\right|^2 dx \tag{1.9}$$

has the property that, there exists $C > 0$ such that

$$\left| \int |u(x)|^2 \, d\mu \right| \leq \epsilon \left\| \nabla u \right\|^2_{L^2(\mathbb{R}^d)} + C\epsilon^{-\beta} \left\| u \right\|^2_{L^2(\mathbb{R}^d)}, \quad \forall u \in C_0^\infty \left(\mathbb{R}^d \right) \tag{1.10}$$

for every $\epsilon > 0$.

(iii): *For μ defined by (1.9),*

$$\lim_{R \to 0+} \sup_{x_0 \in \mathbb{R}^d} \left\{ \frac{\left\| \mu_{B(x_0,R)} \right\|^2_{\dot{H}^{-1}(\mathbb{R}^d)}}{\mu \left(B \left(x_0, R \right) \right)} \right\} = 0,$$

where $\mu_{B(x_0,R)}$ is the restriction of μ to the ball $B(x_0, R)$.

Before proceeding to our main result, it is instructive to demonstrate the cancellation phenomenon mentioned above by considering an example of a strongly oscillating weight.

Example 1. *Let us set*

$$f(x) = |x|^{d-2} \sin \left(|x|^d \right),$$

where $d \geq 3$ is an integer, which may be arbitrary large. Obviously, both f_+ and f_- fail to satisfy (1.3) due to the growth of the amplitude at infinity. However,

$$f(x) = div \overrightarrow{F}(x) + O \left(|x|^{-2} \right), \ \text{where} \ \overrightarrow{F}(x) = -\frac{1}{d} \frac{\overrightarrow{x}}{|x|^2} \cos \left(|x|^d \right). \tag{1.11}$$

By Hardy' s inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx \leq \frac{4}{(d-2)^2} \left\| \nabla u \right\|^2_{L^2}, \qquad u \in C_0^\infty \left(\mathbb{R}^d \right),$$

and hence the term $O\left(|x|^{-2} \right)$ in (1.11) is harmless, whereas $\overrightarrow{F}$ clearly satisfies (1.10) since $\left| \overrightarrow{F}(x) \right|^2 \leq |x|^{-2}$. This shows that f is admissible for (1.2), while $|f|$ is obviously not (see [6]*).*

Theorem 2. *Let f be a complex-valued distribution on $\mathbb{R}^d$, $d \geq 3$ and let $0 < \beta \leq 1$. Then (1.4) holds if and only if f is the divergence of a vector-field $\overrightarrow{F} : \mathbb{R}^d \to \mathbb{C}^d$ such that*

$$\int_{B(x_0,R)} \left| \overrightarrow{F}(x) - m_{B(x_0,R)} \left(\overrightarrow{F} \right) \right|^2 dx \leq const \ R^{d-2+\frac{4}{1+\beta}}, \quad \text{for all } R > 0. \tag{1.12}$$

where the constant is independent of x_0 and R. The vector-field $\overrightarrow{F} \in L^2_{loc} \left(\mathbb{R}^d \right)^d$ can be chosen as $\overrightarrow{F} = \nabla \Delta^{-1} f$ (see [6]*).*

Remark 2.

1.: *In case,* $\beta = 1$, *it follows that (1.12) holds if and only if* $\overrightarrow{F} \in BMO\left(\mathbb{R}^d\right)^d$. *In order,* $f \in BMO^{-1}\left(\mathbb{R}^d\right) = \dot{F}_{\infty}^{-1,2}\left(\mathbb{R}^d\right)$, *where* $\dot{F}_q^{r,p}$ *stands for the scale of homogeneous Triebel-Lizorkin spaces (see* [13]*). Similarly, in the case* $0 < \beta < 1$, *(1.12) holds if and only if* $\overrightarrow{F}$ *is Hölder-continuous :*

$$\left|\overrightarrow{F}(x) - \overrightarrow{F}(y)\right| \leq c\,|x-y|^{\frac{1-\beta}{1+\beta}}, \qquad |x-y| < R.$$

2.: *In the case* $\beta = 1$, *statement (i) of Theorem 2 (sufficiency of the condition* $\overrightarrow{F} \in BMO\left(\mathbb{R}^d\right)^d$*) is equivalent via the* $\mathcal{H}^1 - BMO$ *duality to the inequality*

$$\|u\nabla u\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C\,\|u\|_{L^2(\mathbb{R}^d)}\,\|\nabla u\|_{L^2(\mathbb{R}^d)}, \quad \forall u \in C_0^\infty\left(\mathbb{R}^d\right).$$

Here $\mathcal{H}^1\left(\mathbb{R}^d\right)$ *is the real Hardy space on* $\mathbb{R}^d$ [12]. *The preceding estimate yields the following vector-valued inequality which is used in studies of the Navier-Stokes equation, and is related to the compensated compactness phenomenon (see* [1]*) :*

$$\|(\overrightarrow{u}.\nabla)\,\overrightarrow{u}\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C\,\|\overrightarrow{u}\|_{L^2(\mathbb{R}^d)^d}\,\|\nabla\overrightarrow{u}\|_{L^2(\mathbb{R}^d)^d}$$
$$div\ \overrightarrow{u} = \overrightarrow{0}\ , \quad \forall \overrightarrow{u} \in C_0^\infty\left(\mathbb{R}^d\right)^d.$$

Before proving the theorem, let us established certain localized versions of the necessary condition for (1.4). Set

$$\omega_{R,x_0}(x) = \omega\left(\frac{x-x_0}{R}\right)$$

where $\omega \in C_0^\infty(B(0,1))$ is a smooth cut-off function with the following properties

$$|\omega(x)| \leq 1 \text{ and } |\nabla\omega(x)| \leq 1 \text{ for } x \in B(0,1).$$

With this definition, we obtain the following more useful statement.

Proposition 1. *Suppose* $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$ *and* $0 < \beta \leq 1$. *Suppose that (1.4) holds for every* $R \in (0, +\infty)$. *Let* $\overrightarrow{F}$ *be defined by* $\overrightarrow{F} = \nabla\Delta^{-1} f$.

(a): *For* $d \geq 3$,

$$\int_{\mathbb{R}^d} \left|\nabla\Delta^{-1}\left(\omega_{R,x_0} f\right)\right|^2 dx \leq C R^{d-2+\frac{4}{1+\beta}}\ ,\ 0 < R < +\infty$$

(b): *For* $d \geq 2$,

$$\int\limits_{B(x_0,R)} \left|\nabla\Delta^{-1}\left(\omega_{R,x_0} f\ \right)\right|^2 dx \leq CR^{d-2+\frac{4}{1+\beta}} \quad , \quad 0 < R < +\infty$$

Now we can state the following

Lemma 1. *Suppose* $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$, $d \geq 2$ *and* $0 < \beta \leq 1$. *Suppose that (1.4) holds for every* $R \in (0, +\infty)$. *Then we have*

$$\int\limits_{B(x_0,R)} \left|\nabla\Delta^{-1} f - m_{B(x_0,R)}\left(\nabla\Delta^{-1} f\right)\right|^2 dx \leq C\ R^{d-2+\frac{4}{1+\beta}}.$$

We are now in a position to give the proof of theorem 2. We need only to prove the statement (i) since (ii) follow from Proposition 1 and Lemma 1.

Proof. Suppose that f is represented in the form (1.7) so that (1.12) is satisfied for all $R > 0$. Applying the multiplicative inequality nonnegative measures ([5], th.1.4.7) to $\left|\overrightarrow{F}\right|^2 dx$, we get :

$$\int\limits_{B(x_0,R)} \left|\overrightarrow{F}(x)\right|^2 |u(x)|^2 dx \leq C \left\|\nabla u\right\|_{L^2(\mathbb{R}^d)}^{2\left(\frac{\beta-1}{\beta+1}\right)} \left\|u\right\|_{L^2(\mathbb{R}^d)}^{\frac{4}{\beta+1}}.$$

Hence,

$$\begin{aligned} |< fu, u >| = \left|< \overrightarrow{F}u, \nabla u >\right| &\leq \left\|\overrightarrow{F}u\right\|_{L^2(\mathbb{R}^d)} \left\|\nabla u\right\|_{L^2(\mathbb{R}^d)} \\ &\leq C_1^{\frac{1}{2}} \left\|\nabla u\right\|_{L^2(\mathbb{R}^d)}^{1+\frac{\beta-1}{\beta+1}} \left\|u\right\|_{L^2(\mathbb{R}^d)}^{\frac{2}{\beta+1}} \end{aligned}$$

Combining the preceding estimates with the following inequality ([7], th 3.2.1) :

$$\|u\|_{L^2} \leq C(d)R \left\|\nabla u\right\|_{L^2}, \quad u \in C_0^\infty\left(B\left(x_0, R\right)\right),$$

we get

$$|< fu, u >| \leq CR^{\frac{2}{1+\beta}} \left\|\nabla u\right\|_{L^2}^2, \qquad u \in C_0^\infty\left(B\left(x_0, R\right)\right).$$

The proof of theorem 2 is complete. □

We use know characterizations of the Morrey-Campanato spaces. In particular,

Proposition 2. *For* $0 < \beta < 1$, *condition (1.12) is equivalent to the condition* $\overrightarrow{F} \in \Lambda_\gamma\left(\mathbb{R}^d\right)$ *where* $\gamma = \frac{1-\beta}{1+\beta}$. *In the case* $\beta = 1$, *we have* $\overrightarrow{F} \in BMO\left(\mathbb{R}^d\right)^d$.

It is easy to see that in the case $\beta = 1$, the sufficiently part of Theorem 2 is equivalent to inequality :

$$\left| < \overrightarrow{F} u, \nabla u > \right| \leq C \left\| \overrightarrow{F} \right\|_{BMO(\mathbb{R}^d)^d} \|u\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^2(\mathbb{R}^d)}, \forall u \in C_0^\infty \left(\mathbb{R}^d\right)$$

By duality, the preceding inequality yields :

$$\|u \nabla u\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C \|u\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^2(\mathbb{R}^d)}, \quad \forall u \in C_0^\infty \left(\mathbb{R}^d\right).$$

where $\mathcal{H}^1 \left(\mathbb{R}^d\right)$ is a real Hardy space [12]. Such inequalities are useful in hydrodynamics [1]. As an immediate consequence, we obtain the vector-valued quadratic form :

$$\|(\overrightarrow{u}.\nabla)\overrightarrow{u}\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C \|\overrightarrow{u}\|_{L^2(\mathbb{R}^d)^d} \|\nabla \overrightarrow{u}\|_{L^2(\mathbb{R}^d)^d}$$
$$\text{div } \overrightarrow{u} = \overrightarrow{0}, \forall \overrightarrow{u} \in C_0^\infty \left(\mathbb{R}^d\right)^d.$$

Both of the preceding inequalities are corollaries of the homogeneous version of the "$div - curl$" Lemma [1]. The following corollary which is an immediate consequence of Theorem 2 and the characterizations of Morrey-Campanato spaces [3], gives a necessary and sufficient condition for (1.12) in terms of homogeneous Besov spaces of negative order.

Corollary 2. *Under the assumptions of Theorem 2, in the case $\beta = 1$, condition (1.12) is equivalent to $f \in BMO^{-1}\left(\mathbb{R}^d\right)$. Similarly, in the case $0 < \beta < 1$, condition (1.12) is equivalent to $f \in \dot{B}_\infty^{-\frac{2\beta}{1+\beta},\infty}\left(\mathbb{R}^d\right)$.*

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Received September 1, 2005

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